# Bounded Perturbation Resilience of the Adaptive Projected Subgradient Method

Jochen Fink | Technische Universität Berlin, Fraunhofer Heinrich-Hertz-Institute | 9th Annual Loma Linda Workshop

# Fixed Point Algorithms and Superiorization in Wireless Communication Systems

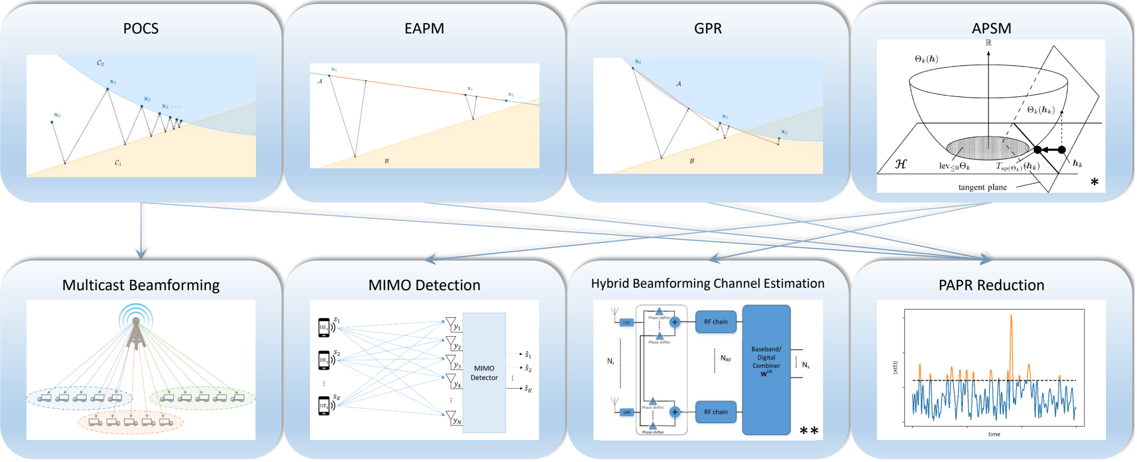


Figure: Overview of methods and applications considered in [Fin22]. (* Source: [YDLY09]; ** Source: [AKS⁺18])

## Definition (Bounded Perturbations)

A sequence $(\beta_n \mathbf{y}_n)_{n\in\mathbb{N}}$ in $\mathcal{H}$ is called a **sequence of bounded perturbations** if $(\beta_n)_{n\in\mathbb{N}} \in \ell_+^1(\mathbb{N})$ and $(\exists r \in \mathbb{R})(\forall n \in \mathbb{N})$ $\|\mathbf{y}_n\| \leq r$.

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– An algorithm defined by a mapping $T : \mathcal{H} \to \mathcal{H}$ is **bounded perturbation resilient**, if convergence to a **fixed point** $\mathbf{x} \in \mathrm{Fix}(T) = \{\mathbf{x} \in \mathcal{H} \mid T(\mathbf{x}) = \mathbf{x}\}$ is guaranteed even if bounded perturbations are added to its iterates.

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- Starting from a **basic algorithm**

$$(\forall n \in \mathbb{N}) \quad \mathbf{x}_{n+1} = T(\mathbf{x}_n), \quad \mathbf{x}_0 \in \mathcal{H},$$

the **superiorization methodology** [CDH10, Cen15] automatically generates its **superiorized version**

$$(\forall n \in \mathbb{N}) \quad \mathbf{x}_{n+1} = T(\mathbf{x}_n + \beta_n \mathbf{y}_n), \quad \mathbf{x}_0 \in \mathcal{H}$$

by defining a sequence $(\beta_n \mathbf{y}_n)_{n\in\mathbb{N}}$ of bounded perturbations (typically based on subgradients of a convex objective function).

# Bounded Perturbation Resilience and Superiorization

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  by defining a sequence $(\beta_n \mathbf{y}_n)_{n\in\mathbb{N}}$ of bounded perturbations (typically based on subgradients of a convex objective function).
- (We also consider basic algorithms defined by a sequence $(T_n)_{n\in\mathbb{N}}$ of mappings)

## Quasi Fejér Monotonicity

**Definition (Quasi-Fejér Monotonicity)**

Let $\mathcal{S}$ be a nonempty subset of $\mathcal{H}$ and let $(\mathbf{x}_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{H}$. Then $(\mathbf{x}_n)_{n\in\mathbb{N}}$ is [Com01]

- **quasi-Fejér (monotone) of Type-I** relative to $\mathcal{S}$ if

$$(\exists(\varepsilon_n)_{n\in\mathbb{N}} \in \ell_+^1(\mathbb{N}))(\forall \mathbf{z} \in \mathcal{S})(\forall n \in \mathbb{N}) \quad \|\mathbf{x}_{n+1} - \mathbf{z}\| \leq \|\mathbf{x}_n - \mathbf{z}\| + \varepsilon_n.$$

- **quasi-Fejér (monotone) of Type-II** relative to $\mathcal{S}$ if

$$(\exists(\varepsilon_n)_{n\in\mathbb{N}} \in \ell_+^1(\mathbb{N}))(\forall \mathbf{z} \in \mathcal{S})(\forall n \in \mathbb{N}) \quad \|\mathbf{x}_{n+1} - \mathbf{z}\|^2 \leq \|\mathbf{x}_n - \mathbf{z}\|^2 + \varepsilon_n.$$

- **quasi-Fejér (monotone) of Type-III** relative to $\mathcal{S}$ if

$$(\forall \mathbf{z} \in \mathcal{S})(\exists(\varepsilon_n)_{n\in\mathbb{N}} \in \ell_+^1(\mathbb{N}))(\forall n \in \mathbb{N}) \quad \|\mathbf{x}_{n+1} - \mathbf{z}\|^2 \leq \|\mathbf{x}_n - \mathbf{z}\|^2 + \varepsilon_n.$$

# General Approach Used in [Fin22]

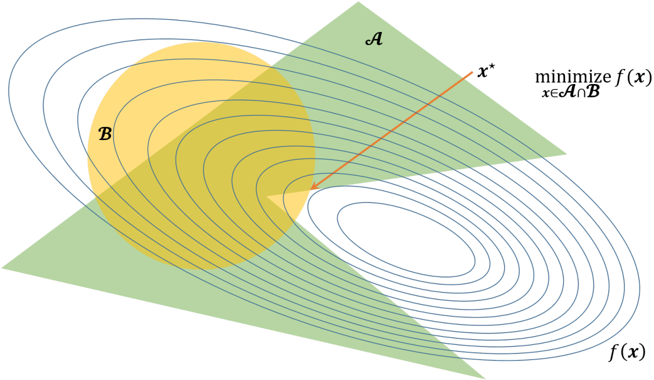


Figure: Original optimization problem.

1. Pose the problem in a Hilbert space

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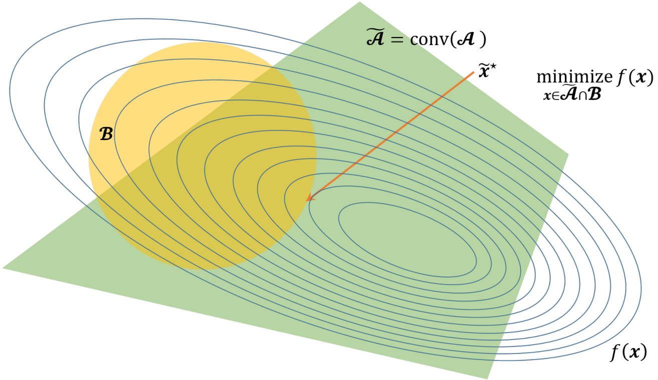


Figure: Relaxed optimization problem.

1. Pose the problem in a Hilbert space
2. Relax nonconvex constraints

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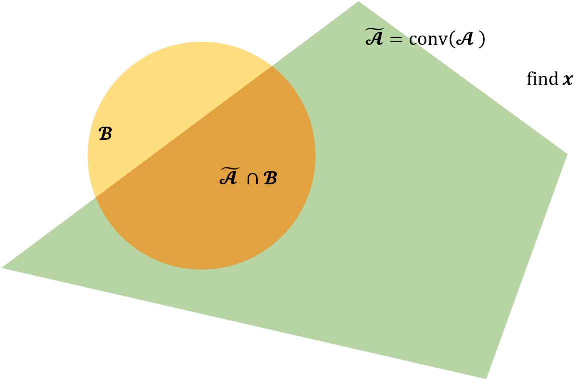


find $\boldsymbol{x} \in \widetilde{\mathcal{A}} \cap \mathcal{B}$

Figure: Convex feasibility problem.

1. Pose the problem in a Hilbert space
2. Relax nonconvex constraints
3. Omit the objective function

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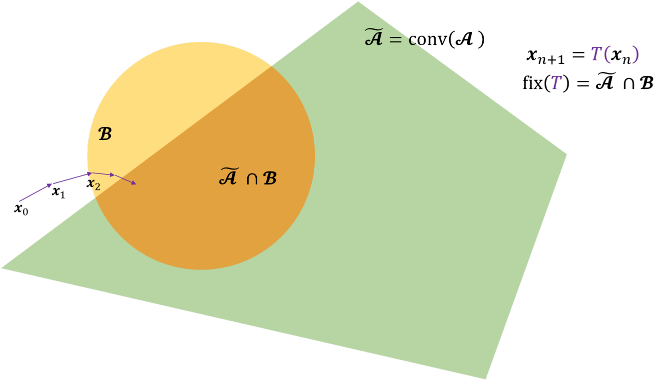


Figure: Fixed point algorithm for the convex feasibility problem.

1. Pose the problem in a Hilbert space
2. Relax nonconvex constraints
3. Omit the objective function
4. Design a bounded perturbation resilient fixed point algorithm for the convex feasibility problem

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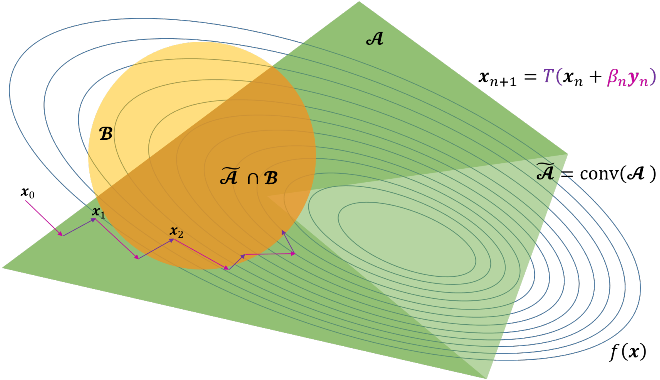


Figure: Superiorized fixed point algorithm.

1. Pose the problem in a Hilbert space
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5. Devise perturbations that reduce the objective value and the distance to the nonconvex constraints

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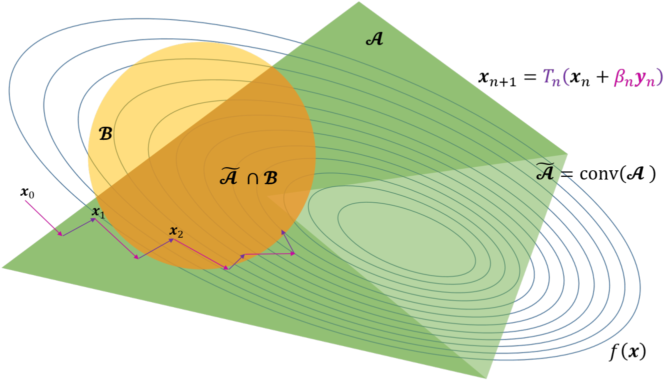


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4. Design a bounded perturbation resilient fixed point algorithm for the convex feasibility problem
5. Devise perturbations that reduce the objective value and the distance to the nonconvex constraints
6. (The generalization to fixed point algorithms defined by a sequence of mappings is also considered)

# Adaptive Projected Subgradient Method (APSM)

The APSM [YO05] extends Polyak's subgradient algorithm [Pol69] to the case where the cost functions change throughout the iterations.

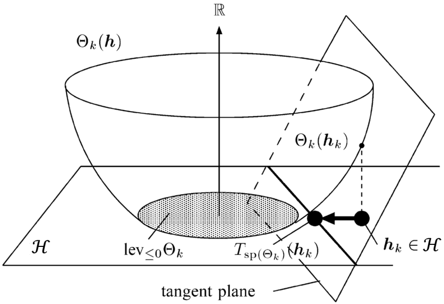


Figure: Illustration of the subgradient projection in an exemplary variant of the APSM. (Source: [YDLY09])

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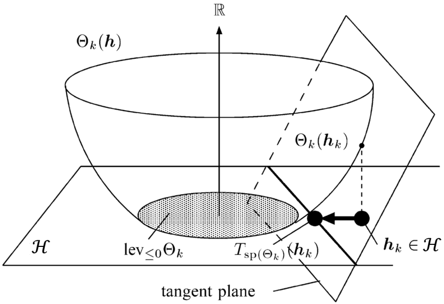


Figure: Illustration of the subgradient projection in an exemplary variant of the APSM. (Source: [YDLY09])

– Aims at minimizing all but finitely many functions of a sequence $(\Theta_n : \mathcal{H} \to \mathbb{R}_+)_{n \in \mathbb{N}}$ continuous convex functions over a closed convex set $\mathcal{K} \subset \mathcal{H}$.

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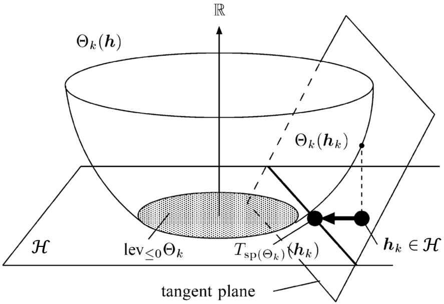


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- Aims at minimizing all but finitely many functions of a sequence $(\Theta_n : \mathcal{H} \to \mathbb{R}_+)_{n\in\mathbb{N}}$ continuous convex functions over a closed convex set $\mathcal{K} \subset \mathcal{H}$.
- Applies the recursion

$$\mathbf{x}_{n+1} = \begin{cases} P_{\mathcal{K}}\left(\mathbf{x}_n - \lambda_n \frac{\Theta_n(\mathbf{x}_n)}{\|\Theta_n'(\mathbf{x}_n)\|^2}\Theta_n'(\mathbf{x}_n)\right) & \text{if } \Theta_n'(\mathbf{x}_n) \neq \mathbf{0}, \\ \mathbf{x}_n & \text{otherwise,} \end{cases}$$

where $\Theta_n'(\mathbf{x}_n) \in \partial\Theta_n(\mathbf{x}_n)$ and $\lambda_n \in [0, 2]$.

# Adaptive Projected Subgradient Method (APSM)

- Can be used to solve convex feasibility problems (possibly in infinite dimensional spaces or with infinitely many constraint sets)
- Practical applications of the APSM include
  - Multiaccess interference suppression [CY08]
  - Acoustic feedback cancellation [YY06, WZQZ10]
  - Robust beamforming [STY09]
  - Robust subspace tracking [CKT14]
  - Online radio-map reconstruction [KCV+15]
  - Kernel-based online classification [STY08]
  - Peak-to-average-power-ratio reduction [CY09]
  - Distributed learning in diffusion networks [CYM09, CST11, SYCD18]
  - Adaptive symbol detection [ACYS18, ACYS20, MMS+22]
- The extrapolated alternating projection method in [BCK06], which has been used for image reconstruction [CCC+12], can be derived as a particular case of the APSM

# Adaptive Projected Subgradient Method (APSM)

– Applications of the superiorized APSM [Fin22]
  - Online channel estimation for hybrid beamforming architectures [FCS20] (Perturbations are used to encourage sparsity)
  - Symbol detection in multi-antenna (MIMO) systems [FCS23] (Perturbations are used to incorporate nonconvex constraints)

## Some results on QNE mappings

**Proposition (Sequences generated by perturbed QNE mappings)**

*Let $(T_n : \mathcal{H} \to \mathcal{H})_{n\in\mathbb{N}}$ be a sequence of quasi-nonexpansive mappings such that $\mathcal{C} := \bigcap_{n\in\mathbb{N}} \mathrm{Fix}(T_n) \neq \emptyset$, and let $(\beta_n \mathbf{y}_n)_{n\in\mathbb{N}}$ be a sequence of bounded perturbations in $\mathcal{H}$. Then the sequence $(\mathbf{x}_n)_{n\in\mathbb{N}}$ generated by*

$$(\forall n \in \mathbb{N}) \quad \mathbf{x}_{n+1} = T_n\left(\mathbf{x}_n + \beta_n \mathbf{y}_n\right), \quad \mathbf{x}_0 \in \mathcal{H},$$

*is quasi-Fejér of Type-I relative to $\mathcal{C}$.*

## Some results on QNE mappings

**Proposition (Sequences generated by perturbed $\kappa$-attracting QNE mappings)**

*Let $\kappa > 0$ and let $(T_n : \mathcal{H} \to \mathcal{H})_{n\in\mathbb{N}}$ be a sequence of $\kappa$-attracting quasi-nonexpansive mappings such that $\mathcal{C} := \bigcap_{n\in\mathbb{N}} \operatorname{Fix}(T_n) \neq \emptyset$, and let $(\beta_n \mathbf{y}_n)_{n\in\mathbb{N}}$ be a sequence of bounded perturbations in $\mathcal{H}$. Then for any bounded subset $\mathcal{U} \subset \mathcal{C}$ the sequence $(\mathbf{x}_n)_{n\in\mathbb{N}}$ generated by*

$$(\forall n \in \mathbb{N}) \quad \mathbf{x}_{n+1} = T_n\left(\mathbf{x}_n + \beta_n \mathbf{y}_n\right), \quad \mathbf{x}_0 \in \mathcal{H}$$

*satisfies the following:* $\left(\exists (\gamma_n)_{n\in\mathbb{N}} \in \ell_+^1(\mathbb{N})\right) (\forall \mathbf{z} \in \mathcal{U})(\forall n \in \mathbb{N})$

$$\left\|\mathbf{x}_{n+1} - \mathbf{z}\right\|^2 \leq \left\|\mathbf{x}_n - \mathbf{z}\right\|^2 - \kappa\left\|\mathbf{x}_{n+1} - \mathbf{x}_n\right\|^2 + \gamma_n.$$

# Theoretical Results

## Theorem (Bounded Perturbation Resilience of the APSM)

*Let* $(\Theta_n : \mathcal{H} \to \mathbb{R}_+)_{n\in\mathbb{N}}$ *be a sequence of continuous convex functions, let* $\mathcal{K} \subset \mathcal{H}$ *be a nonempty closed convex set, and denote the APSM [YO05] update for the nth iteration by*

$$T_n(\mathbf{x}) = \begin{cases} P_{\mathcal{K}}\left(\mathbf{x} - \lambda_n \frac{\Theta_n(\mathbf{x})}{\|\Theta_n'(\mathbf{x})\|^2}\Theta_n'(\mathbf{x})\right) & \text{if } \Theta_n'(\mathbf{x}) \neq \mathbf{0}, \\ P_{\mathcal{K}}(\mathbf{x}) & \text{otherwise,} \end{cases}$$

*where* $\Theta_n'(\mathbf{x}_n) \in \partial\Theta_n(\mathbf{x}_n)$ *and* $\lambda_n \in [0, 2]$*. Moreover, let* $(\beta_n \mathbf{y}_n)_{n\in\mathbb{N}} \subset \mathcal{H}$ *be a sequence of bounded perturbations and suppose that*

$$(\forall n \in \mathbb{N}) \quad \Omega_n := \left\{\mathbf{x} \in \mathcal{K} \;\middle|\; \Theta_n(\mathbf{x}) = \Theta_n^\star := \inf_{\mathbf{x}\in\mathcal{K}} \Theta_n(\mathbf{x})\right\}, \qquad \Theta_n^\star = 0 \quad \text{and} \quad \Omega := \bigcap_{n\in\mathbb{N}} \Omega_n \neq \emptyset.$$

*Then the sequence* $(\mathbf{x}_n)_{n\in\mathbb{N}} \subset \mathcal{K}$ *generated by the perturbed APSM*

$$(\forall n \in \mathbb{N}) \quad \mathbf{x}_{n+1} = T_n\left(\mathbf{x}_n + \beta_n \mathbf{y}_n\right), \quad \mathbf{x}_0 \in \mathcal{K}$$

*satisfies the following:*

(a) *The sequence* $(\mathbf{x}_n)_{n\in\mathbb{N}}$ *is quasi-Fejér monotone of Type-I relative to* $\Omega$*, so* $(\mathbf{x}_n)_{n\in\mathbb{N}}$ *is bounded.*

(b) *Moreover, if* $(\forall n \in \mathbb{N})\ \lambda_n \in [\varepsilon_1, 2 - \varepsilon_2] \subset (0, 2)$*, then* $\lim_{n\to\infty} \Theta_n(\mathbf{x}_n) = 0$*.*

(c) *The conditions for strong convergence of* $(\mathbf{x}_n)_{n\in\mathbb{N}}$ *are the same as for the unperturbed APSM in [YO05].*

## Summary and Main Insights from [Fin22]

- Convergence proofs for perturbed variants of widely used fixed point algorithms
  - POCS, EAPM, EPPM, GPR, APSM
  - Applicable in finite/infinite dimensional real Hilbert spaces
- Superiorized fixed point algorithms can approximate solutions to a wide range of communication problems at low computational cost
  - Perturbations can be used to incorporate nonconvex constraints
  - Proximal mappings (instead of subgradients) are useful to define the perturbations
- Bounded Perturbation Resilience can be useful even in online settings, where information about the solution arrives sequentially

Introduction

Theoretical Results

Summary

## Backup Slides

# Projections onto Convex Sets (POCS)

Let $\mathcal{I} := \{1, \dots, K\}$, let $(\forall k \in \mathcal{I})\ \mathcal{C}_k \subset \mathcal{H}$ be a nonempty closed convex set, and consider the **convex feasibility problem**

$$\text{find } \mathbf{x} \in \mathcal{C}_\star := \bigcap_{k \in \mathcal{I}} \mathcal{C}_k .$$

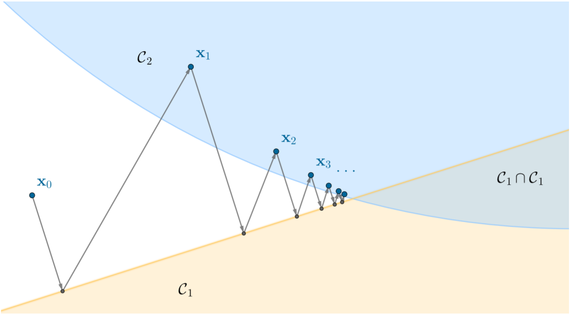



Figure: POCS with two sets and parameters $\lambda_1 = 1$ and $\lambda_2 = 1.3$.

The POCS algorithm produces a sequence $(\mathbf{x}_n)_{n \in \mathbb{N}}$ in $\mathcal{H}$ via

$$(\forall n \in \mathbb{N}) \quad \mathbf{x}_{n+1} = T(\mathbf{x}_n), \quad \mathbf{x}_0 \in \mathcal{H},$$

where $T : \mathcal{H} \to \mathcal{H}$ is given by the composition

$$T := T_{\mathcal{C}_K}^{(\lambda_K)} \cdots T_{\mathcal{C}_1}^{(\lambda_1)},$$

of relaxed projections

$$(\forall \lambda \in [0, 2]) \quad T_{\mathcal{C}}^{(\lambda)} : \mathcal{H} \to \mathcal{H} : \mathbf{x} \mapsto \mathbf{x} + \lambda(P_{\mathcal{C}}(\mathbf{x}) - \mathbf{x})$$

# Extrapolated Alternating Projection Method by Bauschke, Combettes, and Kruk (EAPM)

Let $\mathcal{A} \subset \mathcal{H}$ be a closed affine subspace, let $\mathcal{B} \subset \mathcal{H}$ be nonempty closed convex set, and consider the problem

$$\text{find } \mathbf{x} \in \mathcal{A} \cap \mathcal{B}.$$

The EAPM [BCK06] generates a sequence $(\mathbf{x}_n)_{n\in\mathbb{N}}$ in $\mathcal{A}$ via the recursion

$$(\forall n \in \mathbb{N}) \quad \mathbf{x}_{n+1} = T_\lambda^{\text{EAPM}}(\mathbf{x}_n), \quad \mathbf{x}_0 \in \mathcal{A},$$

where $T_\lambda^{\text{EAPM}} : \mathcal{A} \to \mathcal{A}$ is given by

$$T_\lambda^{\text{EAPM}}(\mathbf{x}) = \mathbf{x} + \lambda K(\mathbf{x}) \left(P_\mathcal{A} P_\mathcal{B}(\mathbf{x}) - \mathbf{x}\right)$$

and $K : \mathcal{A} \to [1, \infty)$ is given by

$$K(\mathbf{x}) = \begin{cases} \frac{\|P_\mathcal{B}(\mathbf{x}) - \mathbf{x}\|^2}{\|P_\mathcal{A} P_\mathcal{B}(\mathbf{x}) - \mathbf{x}\|^2} & \text{if } \mathbf{x} \notin \mathcal{B} \\ 1 & \text{otherwise} \end{cases}$$

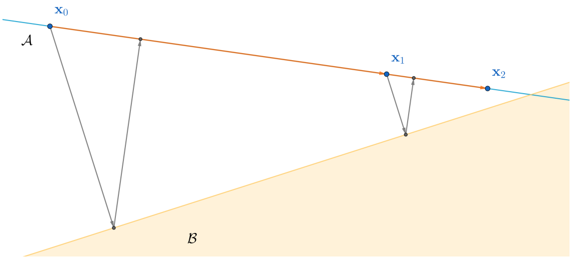


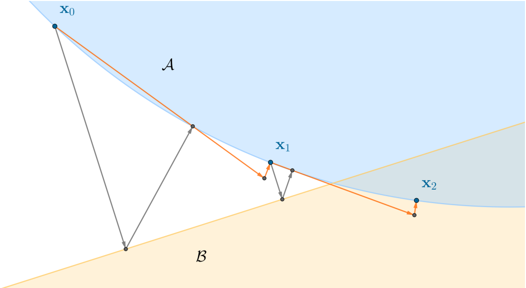

Figure: Illustration of the sequence produced by the EAPM.

# Extrapolated Alternating Projection Method by Gurin, Polyak, and Raik (GPR)

Let $\mathcal{A} \subset \mathcal{H}$ and $\mathcal{B} \subset \mathcal{H}$ be nonempty closed convex sets and consider the convex feasibility problem

$$\text{find } \mathbf{x} \in \mathcal{A} \cap \mathcal{B}.$$

The GPR algorithm [GPR67], [Ceg12, Section 5.2.1.1] generates a sequence $(\mathbf{x}_n)_{n\in\mathbb{N}}$ in $\mathcal{A}$ via the recursion




$$(\forall n \in \mathbb{N}) \quad \mathbf{x}_{n+1} = P_{\mathcal{A}} T^{\mathrm{GPR}}_{\lambda_n}(\mathbf{x}_n), \quad \mathbf{x}_0 \in \mathcal{A},$$

where $T^{\mathrm{GPR}}_{\lambda} : \mathcal{A} \to \mathcal{H}$ is given by

$$T^{\mathrm{GPR}}_{\lambda}(\mathbf{x}) = \mathbf{x} + \lambda\sigma(\mathbf{x})(P_{\mathcal{A}} P_{\mathcal{B}}(\mathbf{x}) - \mathbf{x})$$

and $\sigma : \mathcal{A} \to [1, \infty)$ is given by

$$\sigma(\mathbf{x}) = \begin{cases} \frac{\|P_{\mathcal{B}}(\mathbf{x}) - \mathbf{x}\|^2}{\langle P_{\mathcal{A}} P_{\mathcal{B}}(\mathbf{x}) - \mathbf{x}, P_{\mathcal{B}}(\mathbf{x}) - \mathbf{x}\rangle} & \text{if } \mathbf{x} \notin \mathcal{B} \\ 1 & \text{otherwise.} \end{cases}$$

Figure: Illustration of the sequence produced by the GPR algorithm.

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