The role of superiorization as a tool between feasibility and optimization*

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TRANSFORM METHODS VS FULL DISCRETIZATION OF INVERSE PROBLEMS

$g(x)$ observation

$f(x)$ the casual factors that produced $g(x)$

Froward problem

$f(x)$ is known $\blacktriangleright$ find $g(x)$

$$g(x) = \mathcal{R}(f)(x)$$

Inverse problem

$g(x)$ is known $\blacktriangleright$ find $f(x)$

$$f(x) = \mathcal{R}^{-1}(g)(x)$$

$\swarrow$ $\mathcal{R}$ and $\mathcal{R}^{-1}$ known and tractable $\downarrow$ $\boxed{\text{Transform methods}}$

$\searrow$ otherwise $\downarrow$ Full discretization $\downarrow$ $\boxed{\text{Iterative feasibility-seeking of constraints}}$

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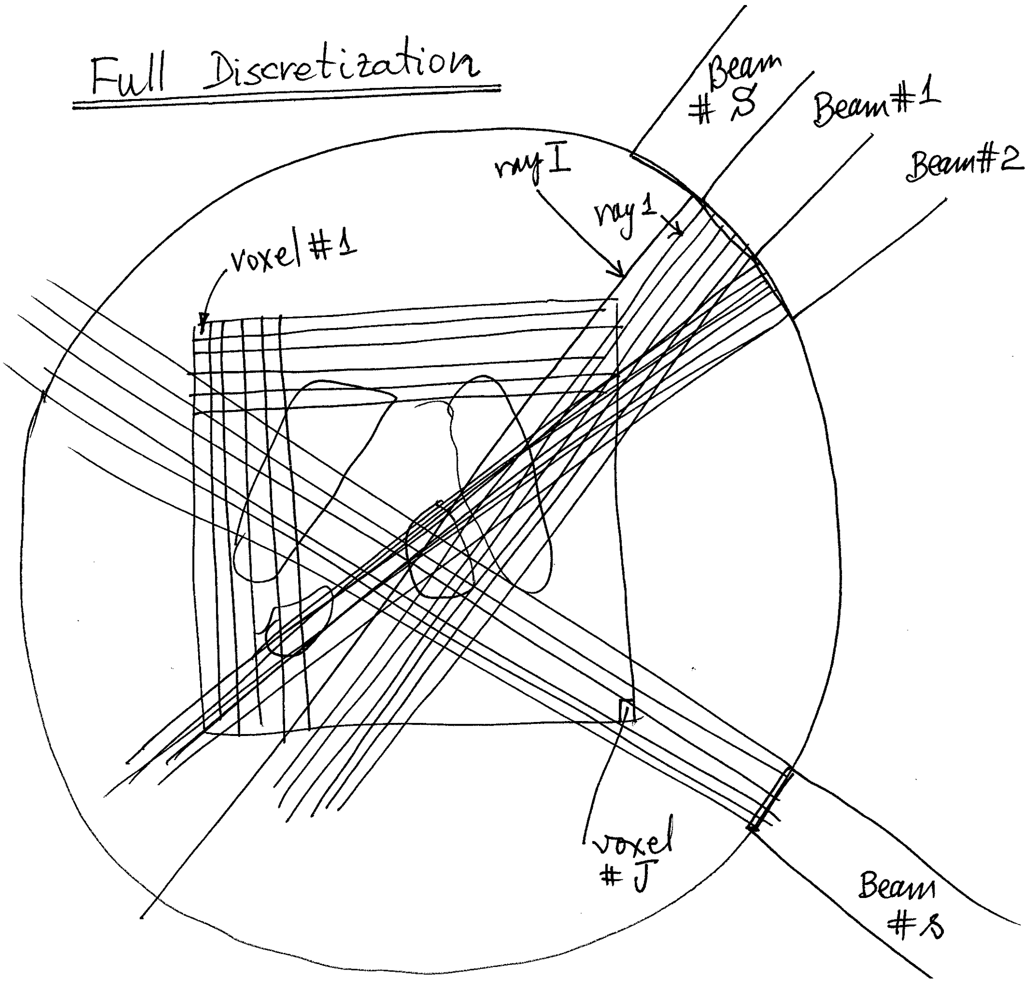


Full discretization: Voxels, beams, rays.

Full discretization of the inverse problem – cont'd.

Prescriptions: For all voxels, indexed $j = 1, 2, \ldots, J$:

· Exact prescriptions:

$$\left\langle a^j, x \right\rangle = b_j$$

· Interval prescriptions:

$$l_j \leq \left\langle a^j, x \right\rangle \leq u_j$$

I The feasibility approach:

$$\begin{cases} \left\langle a^j, x \right\rangle \leq b_l, & \text{for all} \quad j \in B_l, \ l = 1, 2, \ldots, L, \\ t_q \leq \left\langle a^j, x \right\rangle, & \text{for all} \quad j \in T_q, \ q = 1, 2, \ldots, Q, \\ \left\langle a^j, x \right\rangle \leq c, & \text{for all} \quad j \in C, \\ x_i \geq 0, & \text{for all} \quad i = 1, 2, \ldots, I. \end{cases}$$

·

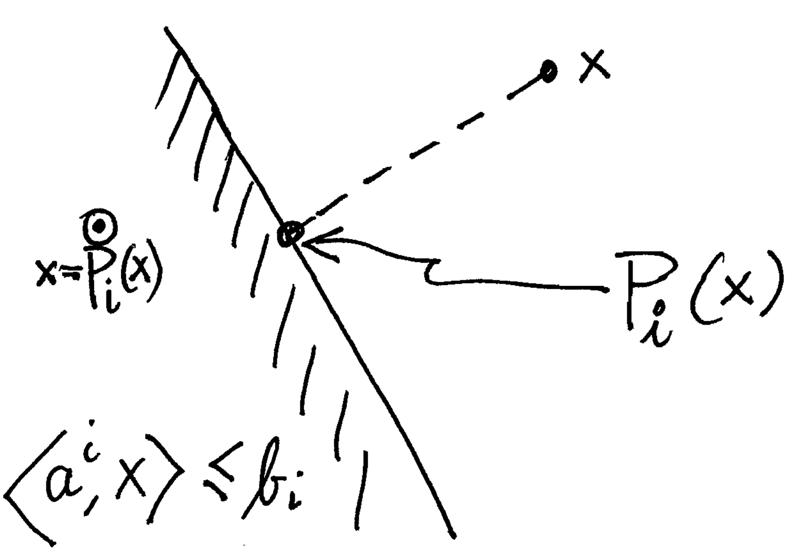
(1) $\langle a^i, x\rangle \le b_i, \quad i = 1, 2, \ldots, I$

(2) $$\min_x \sum_{i=1}^{I} \left(\langle a^i, x\rangle - b_i\right)_+$$

(3) $$\left(\langle a^i, x\rangle - b_i\right)_+ = \begin{cases} 0, & \text{if } \langle a^i, x\rangle - b_i \le 0, \\ \langle a^i, x\rangle - b_i, & \text{if } \langle a^i, x\rangle - b_i > 0. \end{cases}$$

(4) $$\min_x \frac{1}{2} \sum_{i=1}^{I} \left(\left(\langle a^i, x\rangle - b_i\right)_+\right)^2$$

(5) $$\min_x \frac{1}{2} \sum_{i=1}^{I} w_i \left(\left(\langle a^i, x\rangle - b_i\right)_+\right)^2$$



(6)

$$\min_x \frac{1}{2} \sum_{i=1}^{I} \lambda_i \left\| P_i(x) - x \right\|^2 \qquad (7)$$

Fact 1: Up to weights adjustments

(8) $$\frac{1}{2}\sum_{i=1}^{I}\lambda_i\|P_i(x)-x\|^2 = \frac{1}{2}\sum_{i=1}^{I} w_i\left(\left(\langle a^i,x\rangle - b_i\right)_+\right)^2$$

(9) Fact 2: Applying gradient descent method for the unconstrained minimization of (8) yields precisely the Cimmino simultaneous projections algorithm!

(10) Question: Can we do unconstrained minimization of another proximity function to obtain other projection methods?

(11) Answer: No!

Baillon, Combettes & Cominetti (2012)

Projection Methods

Basic property: To reach any goal that is related to the whole family of sets by performing projections onto the individual sets.

Basic ability: To handle huge-size problems whose dimensions are beyond the capabilities of current, more sophisticated, methods.

Various Algorithmic Structures of Projection Methods:

- Sequential
- Simultaneous
- Block-Iterative (ordered subsets)
- String-Averaging

Different “Projection” Operators:

- Orthogonal projections
- Entropic projections
- Bregman projections (2 kinds)
- Csiszar divergences
- Subgradient projections

Solving Problems:

- Convex feasibility (consistent, inconsistent)
- Split feasibility
- Multiple-sets split feasibility
- Best approximation from point to a family of sets
- Bregman function minimization

Proven Performance in Applications, including, but not limited to:

- Medical Imaging, Electron Microscopy, Cristallography
- Radiation Therapy Treatment Planning
- Machine Learning

The string averaging algorithmic structure

For $t = 1, 2, \ldots, M,$ let $I_t = (i_1^t, i_2^t, \ldots, i_{m(t)}^t),$

be an ordered subset of $\{1, 2, \ldots, m\}$

$$x^0 \in S,$$

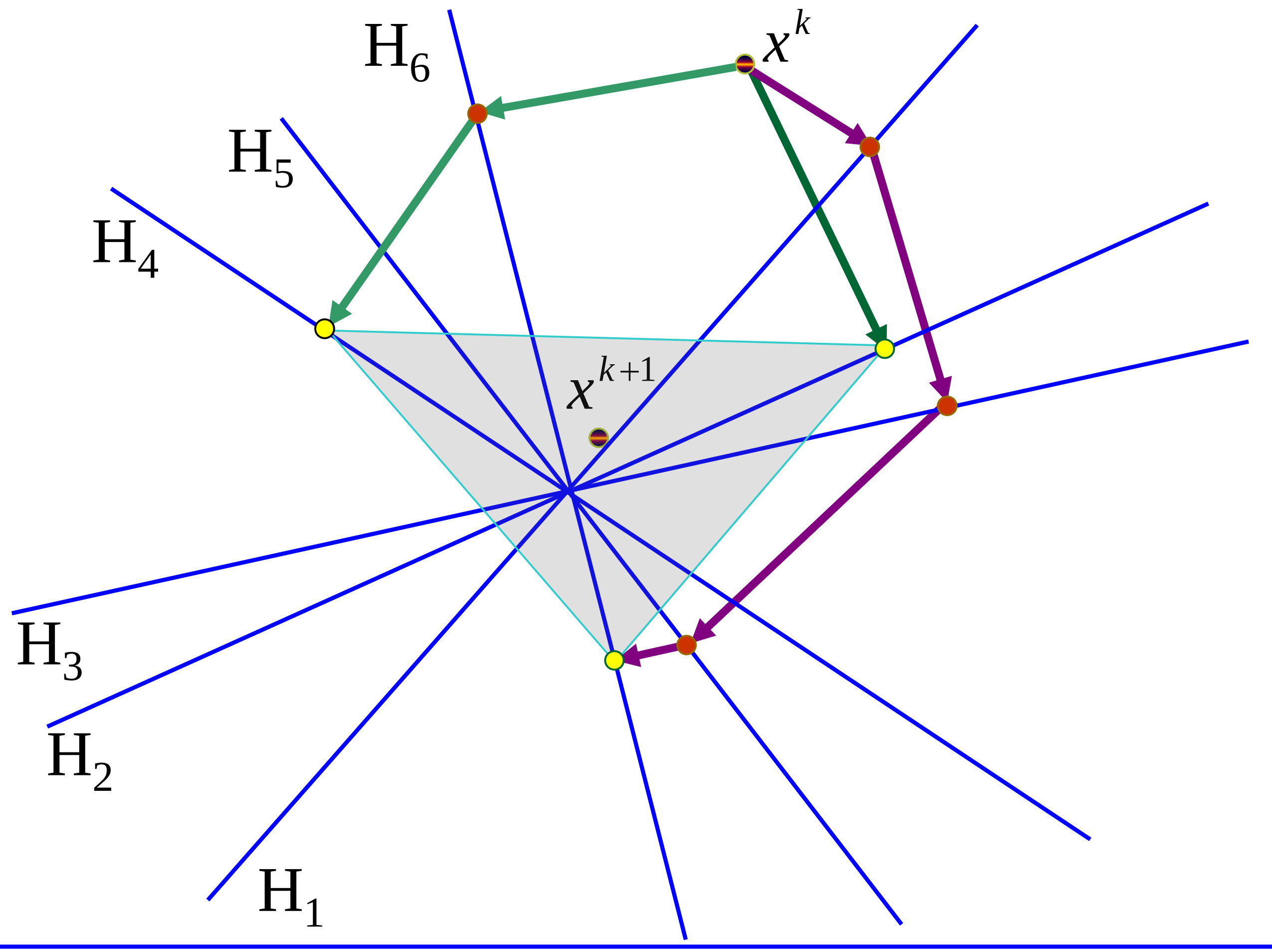
$$T_t x^k = R_{i_{m(t)}^t} \ldots R_{i_2^t} R_{i_1^t} x^k,$$

$$x^{k+1} = R(T_1 x^k, T_2 x^k, \ldots, T_M x^k).$$

Censor, Elfving and Herman, 2001

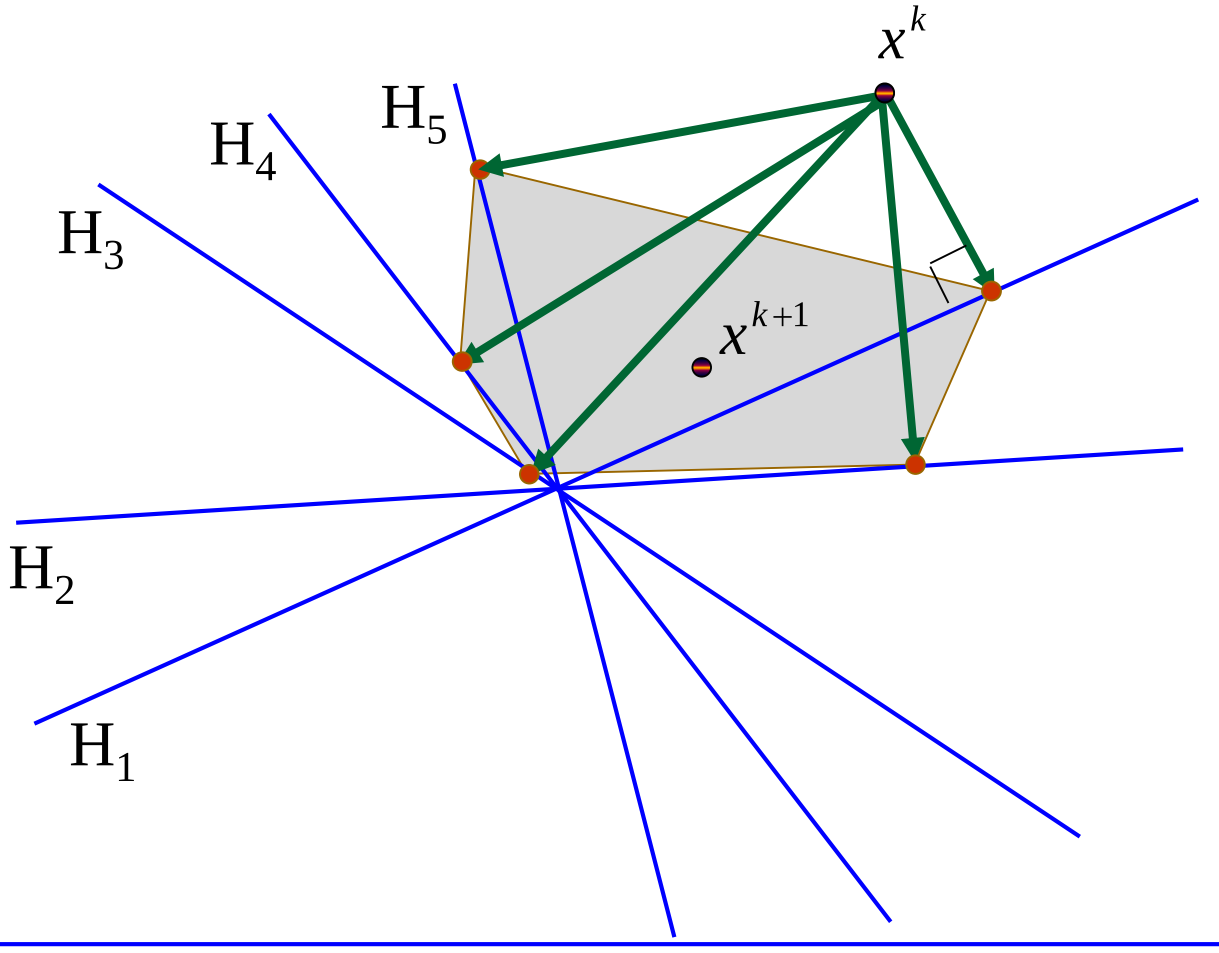
String averaging in general

Strings: $I_1 = (1,3,5,6) \quad I_2 = (2) \quad I_3 = (6,4)$



For example, if all the sets are hyperplanes…

$$\lambda_k = 1 \qquad x^{k+1} = x^k + \lambda_k \sum_{i=1}^{m} w_i (P_i(x^k) - x^k),$$



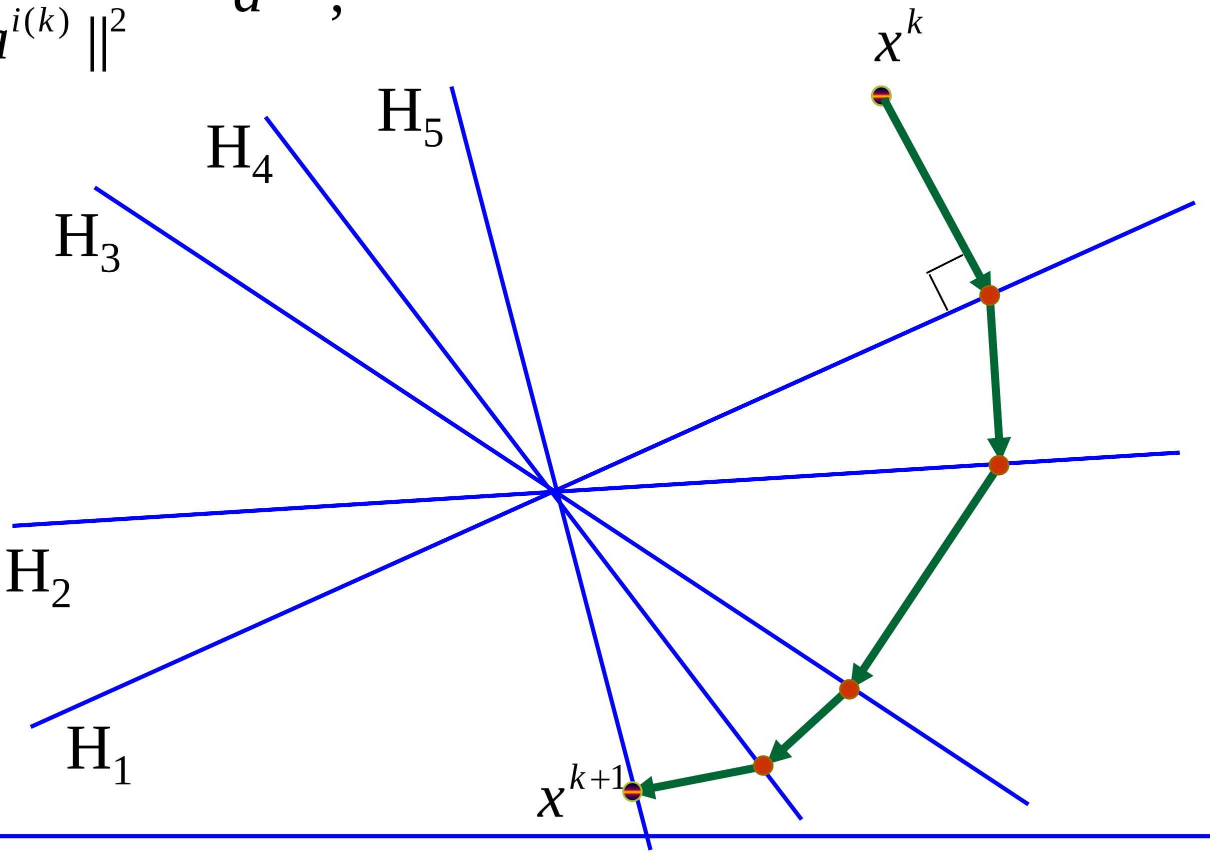
Cimmino 1937

Sequential Successive Projections (POCS, ART, Kaczmarz, Row-Action)

$$x^{k+1} = x^k + \lambda_k (P_{C_{i(k)}}(x^k) - x^k), \; \lambda_k = 1$$

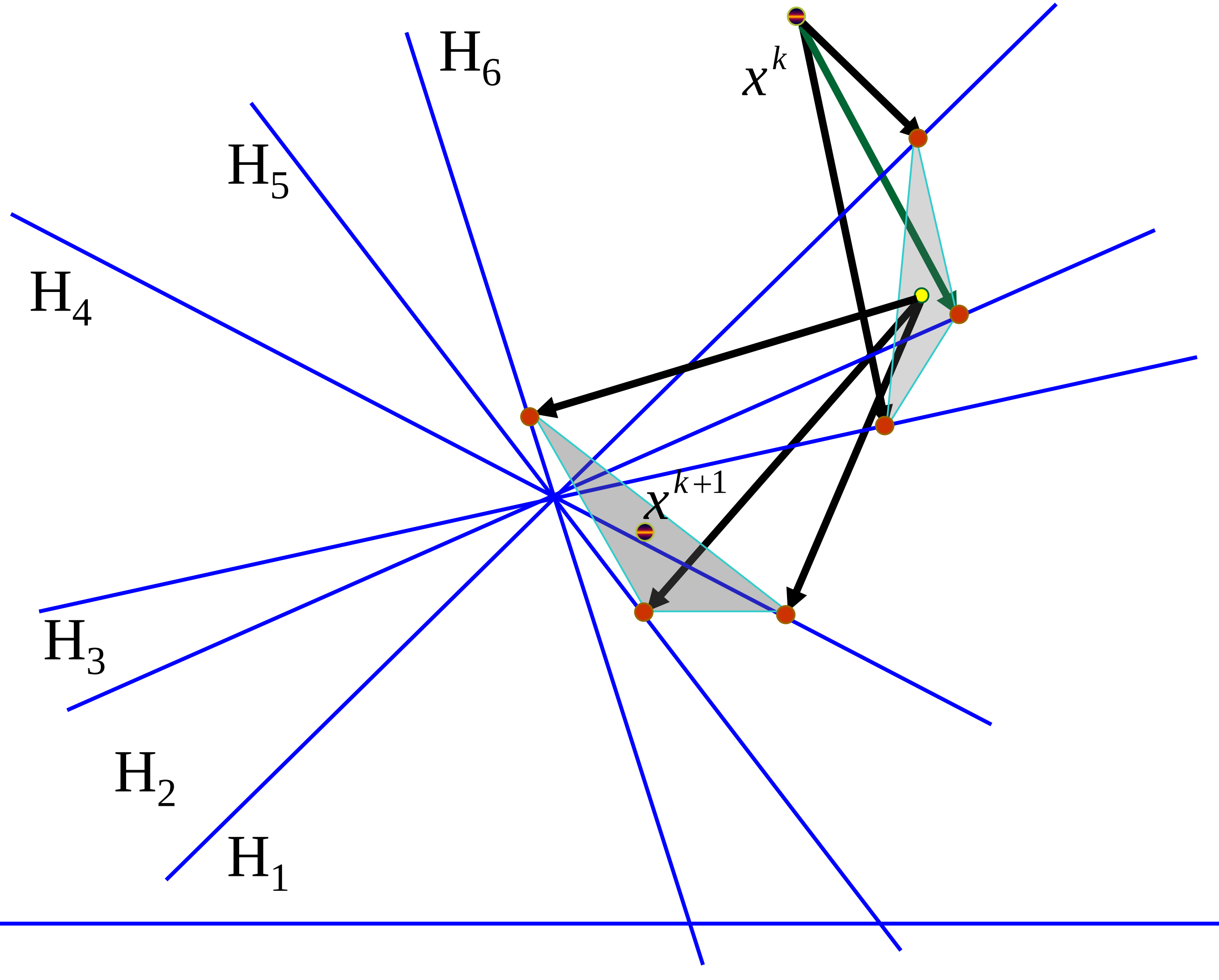
$$\langle a^i, x \rangle - b_i = 0, \quad i = 1, 2, \ldots, m,$$

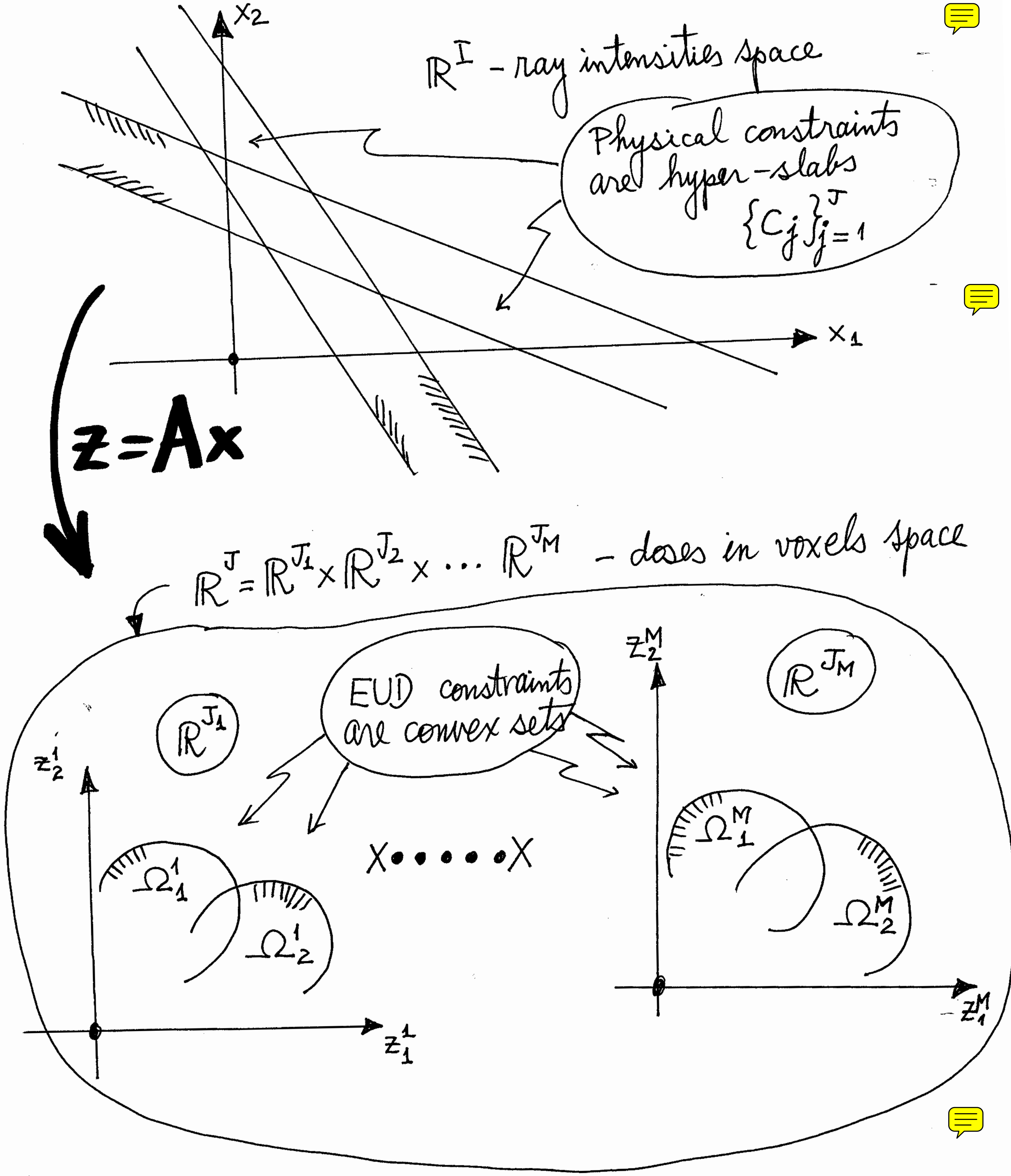
$$x^{k+1} = x^k + \lambda_k \frac{b_{i(k)} - \langle a^{i(k)}, x^k \rangle}{\| a^{i(k)} \|^2} a^{i(k)},$$



Block iterative projections (BIP)

Blocks: $B_1 = (1, 2, 3) \quad B_2 = (4, 5, 6)$




x_2
x_1
$\mathbb{R}^I$ - ray intensities space
Physical constraints are hyper-slabs $\{C_j\}_{j=1}^{J}$
$z = Ax$
$\mathbb{R}^J = \mathbb{R}^{J_1} \times \mathbb{R}^{J_2} \times \cdots \mathbb{R}^{J_M}$ - doses in voxels space
EUD constraints are convex sets
$\mathbb{R}^{J_1}$
$\mathbb{R}^{J_M}$
z_2^1
z_1^1
Ω_1^1
Ω_2^1
X • • • • • X
z_2^M
z_1^M
Ω_1^M
Ω_2^M

The physical model gives rise to constraints such as

$$x \in C_i \quad i = 1, 2, \ldots, m, \qquad C_i \subseteq \mathbb{R}^n$$

Feasibility problem:

Find $x^* \in C := \bigcap_{i=1}^{m} C_i$

Feasibility seeking

minimize a proximity function Prox(x)

unconstrained minimization

$$\begin{cases} \min \operatorname{Prox}(x) \\ x \in \mathbb{R}^n \end{cases}$$

$C \neq \emptyset$ not necessary

Employ feasibility-seeking algorithms

e.g.,

ART

SART

DSAP

BIP

......

many more...

Impose an exogeneous objective function $f(x)$

Constrained minimization:

$$\begin{cases} \min f(x) \\ \text{s.t. } x \in C = \bigcap_{i=1}^{m} C_i \neq \emptyset \end{cases}$$

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Constrained minimization

$$\begin{cases} \min & f(x) \\ \text{s.t.} & x \in C = \bigcap_{i=1}^{m} C_i \end{cases}$$

Regularization

$$\begin{cases} \min \left(f(x) + \gamma\, \mathrm{prox}(x) \right) \\ x \in \mathbb{R}^n \end{cases}$$

Non-regularization

Interior initialization $x^0 \in C$

Barrier methods (interior point methods)

Penalty methods

Exterior initialization $x^0 \in \mathbb{R}^n$

E.g., the projected gradient method (PGM)

This is a good point to tell about Superiorization

Constrained minimization

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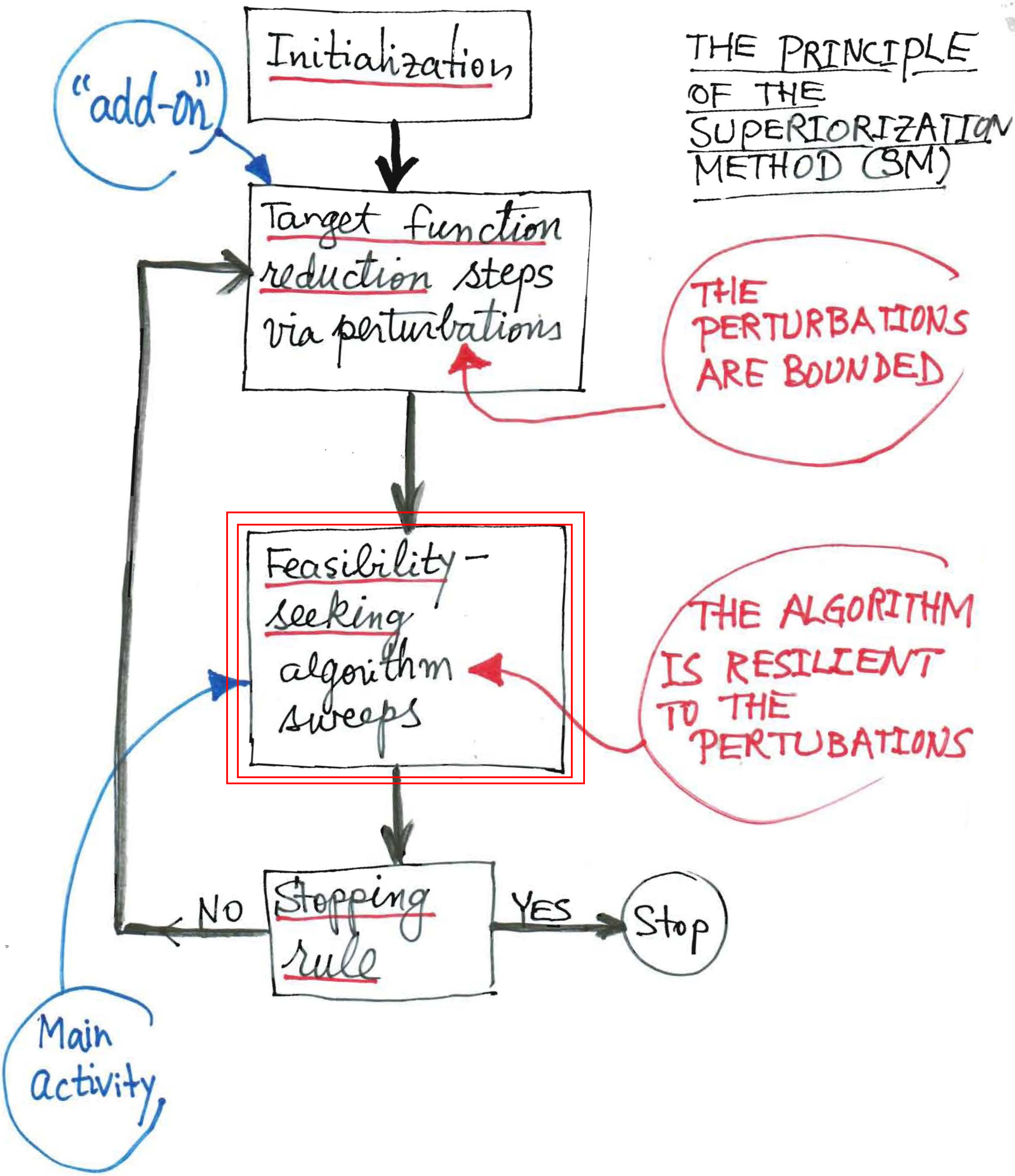
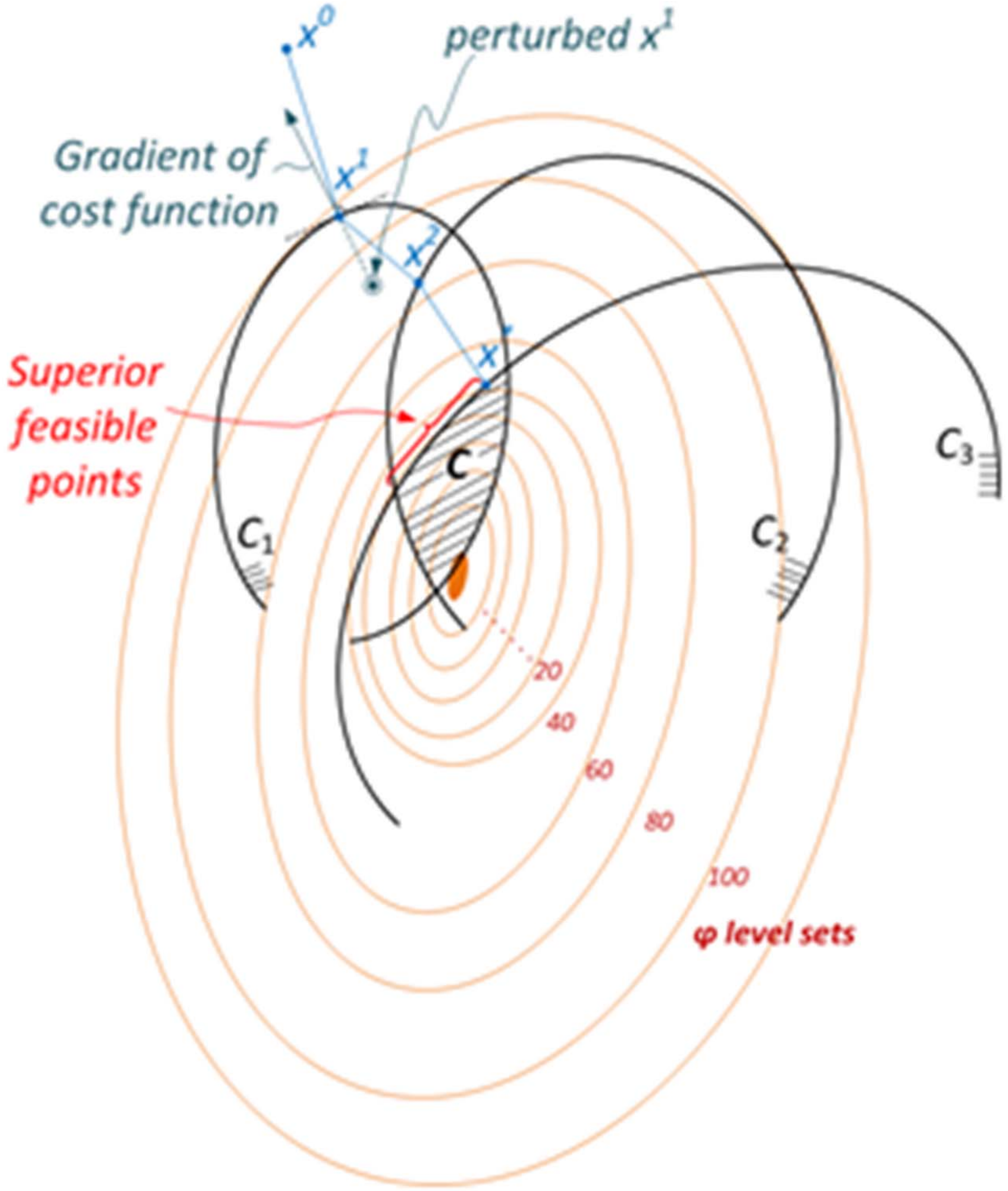
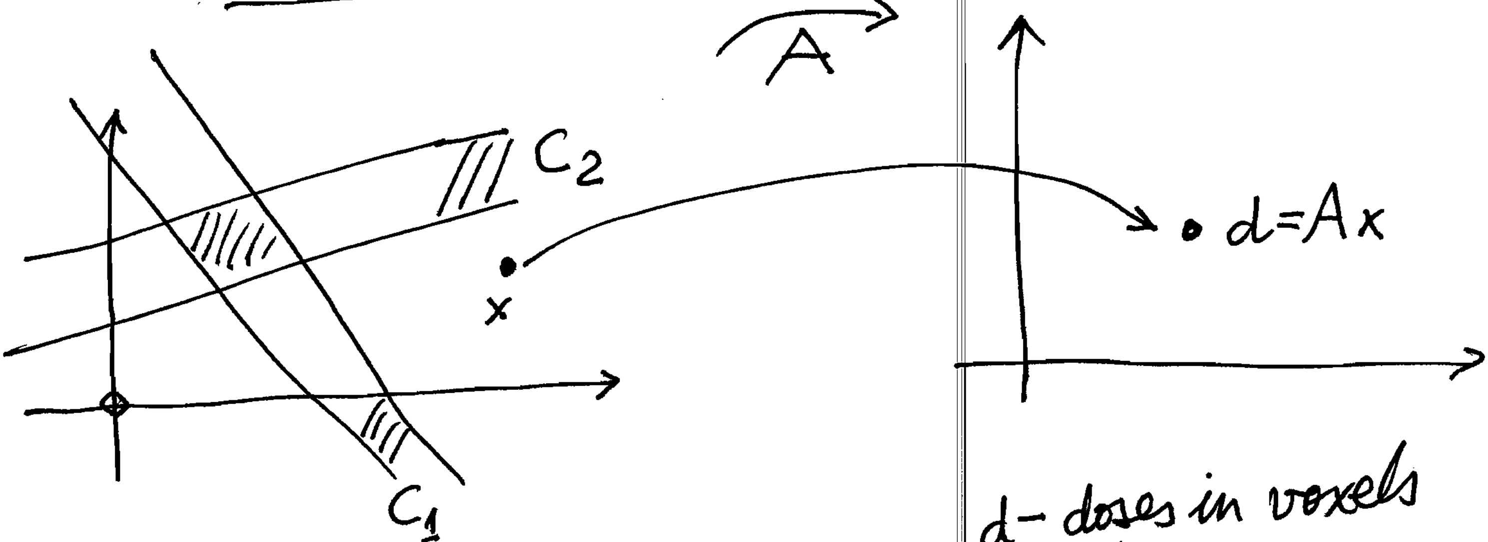

THE PRINCIPLE OF THE SUPERIORIZATION METHOD (SM)
Initialization
"add-on"
Target function reduction steps via perturbations
THE PERTURBATIONS ARE BOUNDED
Feasibility-seeking algorithm sweeps
THE ALGORITHM IS RESILIENT TO THE PERTUBATIONS
Stopping rule
NO
YES
Stop
Main activity

Superiorization Diagram

- C is the feasible set defined by the intersection of many convex sets C_i
- ϕ is a target function to reduce (here not to minimize)
- Systematically perturb (add perturbation term) intermediate iterates from iterative projections in the direction of the negative gradient of ϕ
- Leads to feasible solution that is "superior" to one found without perturbations



TV superiorization in IMRT/IMPT



x – intensities of beamlets vector.

d – doses in voxels vector.

TV Superiorization problems:

Find $x \in \bigcap_{i=1}^{I} C_i$, $x \geq 0$ s.t.

problem I: $TV(x)$ is superior

problem II: $TV(d)$ is superior

problem III: $(TV(d^1), TV(d^2), \ldots, TV(d^S))$ are superior

where

$$d = \begin{pmatrix} d^1 \\ d^2 \\ \vdots \\ d^S \end{pmatrix}$$

subvectors of d for different structures

The Projected Subgradient Minimization (PSM) method

• C is nonempty closed convex set and ϕ is a convex function with domain C.

$$\boxed{x^{k+1} = P_C\left(x^k - t_k \phi'(x^k)\right)}$$

• step-sizes $t_k > 0$, $\phi'(x^k) \in \partial\phi(x^k)$, and P_C is the projection onto C.

• Compare with **The Superiorization Methodology (SM)**

$$\boxed{x^{k+1} = \boldsymbol{P}_T\left(x^k + \beta_k v^k\right)}$$

where $\beta_k v^k$ are bounded perturbations.

• **Underlying philosophy of PSM**: perform unconstrained objective function descent steps via $z^k := x^k - t_k \phi'(x^k)$ and repeatedly **regain feasibility** by doing a projection $P_C\left(z^k\right)$ onto C.

5 Figures

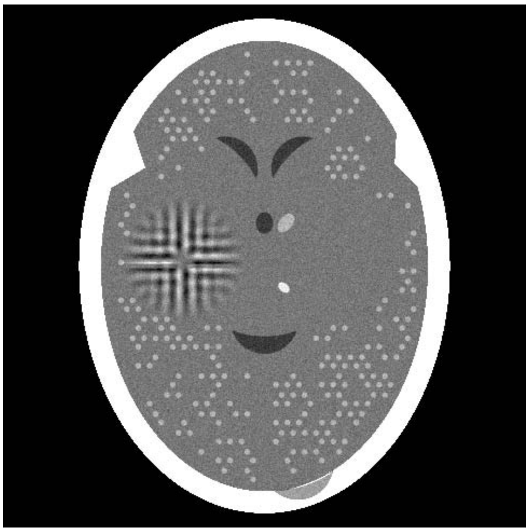


Figure 1: The head phantom. Its tomographic data was obtained for 60 views. It has TV=984.

Figure 2: The image reconstructed by the projected subgradient method (PSM) has TV=919 and was obtained after 5257 seconds.

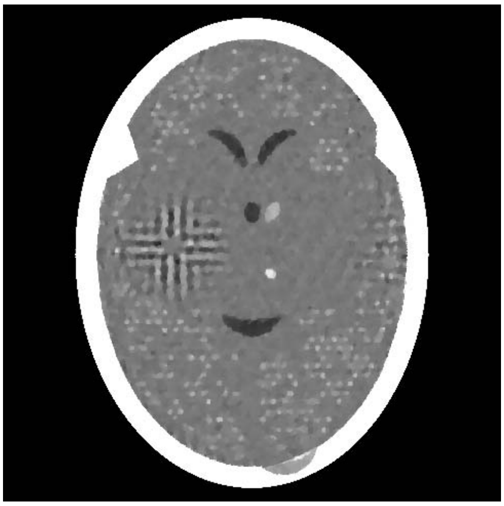


Figure 3: The image reconstructed by the superiorization method has TV=873 and was obtained after 318 seconds.

We explain now what we see in these figures. All computational work was done on a single machine, an Intel i5-3570K 3.4Ghz with 16GB RAM using the SNARK09 software package [27]; the phantom, the data, the reconstructions and displays were all generated within this same framework. In particular, this implies that differences in the reported reconstruction times are not due to the

- $v^{k,n}$: normalized non-ascending perturbation vector for ϕ at $x^{k,n}$, i.e.,

$$v^{k,n} = -\frac{\nabla\phi(x^{k,n})}{\left|\left|\nabla\phi(x^{k,n})\right|\right|} = \phi'(x^{k,n})$$

.

- P_T : projection operator representative of an iterative feasibility-seeking algorithm.

Appendix B
NTVS Algorithm

A pseudocode definition of the NTVS algorithm is written as follows:

set $k = 0$
set $\ell_{-1} = 0$
set $x^k = \bar{x}$
while $k < K$ **do**
    set $n = 0$
    set $\ell_k = \text{rand}(k, \ell_{k-1})$
    set $x^{k,n} = x^k$
    while $n < N$ **do**
        set $v^{k,n} = \phi'(x^{k,n})$
        set $\beta_{k,n} = \alpha^{\ell_k}$
        set $x^{k,n+1} = x^{k,n} + \beta_{k,n} v^{k,n}$
        set $n = n + 1$
        set $\ell_k = \ell_k + 1$
    end while
    set $x^{k+1} = P_T(x^{k,N})$
    set $k = k + 1$
end while

***FOR REFERENCE ONLY**: A pseudocode definition of the NTVS algorithm with the TV reduction requirement included:

set $k = 0$
set $\ell_{-1} = 0$
set $x^k = \bar{x}$
while $k < K$ **do**
    set $n = 0$
    set $\ell_k = \text{rand}(k, \ell_{k-1})$
    set $x^{k,n} = x^k$
    while $n < N$ **do**
        set $v^{k,n} = \phi'(x^{k,n})$
        set $\beta_{k,n} = \alpha^{\ell_k}$
        set $loop = true$
        while $loop$ **do**
            set $z^{k,n} = x^{k,n} + \beta_{k,n} v^{k,n}$
            if $\phi(z^{k,n}) \leq \phi(x^{k,n})$ **then**
                set $x^{k,n} = z^{k,n}$
                set $loop = false$
            end if
            set $\ell_k = \ell_k + 1$
        end while
        set $n = n + 1$
    end while
    set $x^{k+1} = P_T(x^{k,N})$
    set $k = k + 1$
end while

Acknowledgment

The authors greatly appreciate the constructive comments of three anonymous reviewers which helped them to significantly improve this paper. The content of this paper is solely the responsibility of the authors and does not necessarily represent the official views of NBIB or NIH.

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Superiorization and Perturbation Resilience of Algorithms: A Bibliography compiled and continuously updated by Yair Censor

This page (at: http://math.haifa.ac.il/yair/bib-superiorization-censor.html) is a, chronologically ordered, bibliography of scientific publications on the superiorization methodology and perturbation resilience of algorithms, compiled and continuously updated by Yair Censor. If you know of a related work in any form (preprint, reprint, journal publication, conference report, abstract or poster, book chapter, thesis, etc.) that should be included here kindly write to me on: yair@math.haifa.ac.il with full bibliographic details, a DOI if available, and a PDF copy of the work if possible.

This page was initiated on March 7, 2015, and has been last updated on April 16, 2020.
Back to Yair Censor's homepage.

Alphabetically ordered list of authors whose work(s) on, or related to, Superiorization and Perturbation Resilience of Algorithms is cited on this Internet page. Correct as of: March 27, 2020. The List, 288KB, PDF

Down to bottom of page

Trailer:

We replace the text that appeared in this trailer in the previous versions of the page with a quotation of three paragraphs from the preface to the special issue: Y. Censor, G.T. Herman and M. Jiang (Guest Editors), "Superiorization: Theory and Applications", Special Issue of the journal Inverse Problems, Volume 33, Number 4, April 2017 [50] (all references refer to the bibliography below), followed by some additional notes.

"The superiorization methodology is used for improving the efficacy of iterative algorithms whose convergence is resilient to certain kinds of perturbations. Such perturbations are designed to 'force' the perturbed algorithm to produce more useful results for the intended application than the ones that are produced by the original iterative algorithm. The perturbed algorithm is called the 'superiorized version' of the original unperturbed algorithm. If the original algorithm is computationally efficient and useful in terms of the application at hand and if the perturbations are simple and not expensive to calculate, then the advantage of this method is that, for essentially the computational cost of the original algorithm, we are able to get something more desirable by steering its iterates according to the designed perturbations. This is a very general principle that has been used successfully in some important practical applications, especially for inverse problems such as image reconstruction from projections, intensity-modulated radiation therapy and nondestructive testing, and awaits to be implemented and tested in additional fields.

An important case is when the original algorithm is 'feasibility-seeking' (in the sense that it strives to find some point that is compatible with a family of constraints) and the perturbations that are introduced into the original iterative algorithm aim at reducing (not necessarily minimizing) a given merit function. In this case superiorization has a unique place in optimization theory and practice. Many constrained optimization methods are based on methods for unconstrained optimization that are adapted to deal with constraints. Such is, for example, the class of projected gradient methods wherein the unconstrained minimization inner step 'leads' the process and a projection onto the whole constraint set (the feasible set) is performed after each minimization step in order to regain feasibility. This projection onto the constraints set is in itself a non-trivial optimization problem and the need to solve it in every iteration hinders projected gradient methods and limits their efficiency only to feasible sets that are 'simple to project onto.' Barrier or penalty methods likewise are based on unconstrained optimization combined with various 'add-on's that guarantee that the constraints are preserved. Regularization methods embed the constraints into a 'regularized' objecive function and proceed with unconstrained solution methods for the new regularized objective function.

In contrast to these approaches, the superiorization methodology can be viewed as an antipodal way of thinking. Instead of adapting unconstrained minimization algorithms to handling constraints, it adapts feasibility-seeking algorithms to reduce merit function values. This is done while retaining the feasibility-seeking nature of the algorithm and without paying a high computational price. Furthermore, general-purpose approaches have been developed for automatically superiorizing iterative algorithms for large classes of constraints sets and merit functions; these provide algorithms for many application tasks." (end of qoute.)

To a novice on the superiorization methodology and perturbation resilience of algorithms we recommend to read first the recent reviews in [16, 25, 39]. For a recent description of previous work that is related to superiorization but is not included here, such as the works of Sidky and Pan, e.g., [6], we direct the reader to [24, section 3]. The SNARK14 software package [42], with its in-built capability to superiorize iterative algorithms to improve their performance, can be helpful to practitioners. Naturally there is variability among the bibliography items below in their degree of relevance to the superiorization methodology and perturbation resilience of algorithms. In some, such as in, e.g., [23] below, superiorization does not appear in the title, abstract or introduction but only inside the work, e.g., [23, Subsection 6.2.1: Optimization vs. Superiorization].

A word about the history. The terms and notions "superiorization" and "perturbation resilience" first appeared in the 2009 paper of Davidi, Herman and Censor [7] which followed its 2007 forerunner by Butnariu, Davidi, Herman and Kazantsev [3]. The ideas have some of their roots in the 2006 and 2008 papers of Butnariu, Reich and Zaslavski [2, 4]. All these culminated in Ran Davidi's 2010 Ph.D. dissertation [13].

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