Quaternions for Rotation and Orientation: An Overview

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History

History

- Euler - 1748

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- Olinde Rodrigues - 1840

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- Gauss - 1819
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Quaternions

$$a \quad + \quad b\mathbf{i} \quad + \quad c\mathbf{j} \quad + \quad d\mathbf{k}$$

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$$

Quaternions

a	$+$	$b\mathbf{i}$	$+$	$c\mathbf{j}$	$+$	$d\mathbf{k}$
scalar			vector			
real			imaginary			
w		x		y		z

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Quaternions

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$$\mathbb{H} \supset \mathbb{C} \supset \mathbb{R} \supset \mathbb{Q} \supset \mathbb{Z} \supset \mathbb{N}$$

Multiplication

Scalar multiplication

$$a\mathbf{q} = \mathbf{q}a$$

- commutative

Multiplication

Basis multiplication — multiplication of **i**, **j**, **k**.

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	1	**i**	**j**	**k**
1	1	**i**	**j**	**k**
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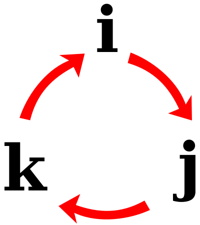

Multiplication

Quaternion multiplication (Hamilton multiplication)

$$(a_1 + b_1\mathbf{i} + c_1\mathbf{j} + d_1\mathbf{k})(a_2 + b_2\mathbf{i} + c_2\mathbf{j} + d_2\mathbf{k}) =$$

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$$\begin{array}{ccccccc}
a_1a_2 & + & a_1b_2\mathbf{i} & + & a_1c_2\mathbf{j} & + & a_1d_2\mathbf{k} \\
+b_1a_2\mathbf{i} & + & b_1b_2\mathbf{i}^2 & + & b_1c_2\mathbf{ij} & + & b_1d_2\mathbf{ik} \\
+c_1a_2\mathbf{j} & + & c_1b_2\mathbf{ji} & + & c_1c_2\mathbf{j}^2 & + & c_1d_2\mathbf{jk} \\
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\end{array}$$

$$\begin{array}{ccccccc}
(a_1a_2 & - & b_1b_2 & - & c_1c_2 & - & d_1d_2) \\
(a_1b_2 & + & b_1a_2 & + & c_1d_2 & - & d_1c_2)\mathbf{i} \\
(a_1c_2 & - & b_1d_2 & + & c_1a_2 & - & d_1b_2)\mathbf{j} \\
(a_1d_2 & + & b_1c_2 & - & c_1b_2 & - & d_1a_2)\mathbf{k}
\end{array}$$

Multiplication

Vector form

$$(s_1 + \mathbf{v_1})(s_2 + \mathbf{v_2}) = (s_1 s_2 - \mathbf{v_1} \cdot \mathbf{v_2}) + (s_1 \mathbf{v_2} + s_2 \mathbf{v_1} + \mathbf{v_1} \times \mathbf{v_2})$$

Inverses and Division

For complex numbers,

$$\begin{aligned} p &= (a + b\mathbf{i}), \\ ||p||^2 &= (a^2 + b^2) \\ p^* &= (a - b\mathbf{i}) \end{aligned}$$

So

$$p^{-1} = p^* / ||p||^2 = (a + b\mathbf{i})^{-1} = (a - b\mathbf{i})/(a^2 + b^2)$$

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Likewise for quaternions,

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Likewise for quaternions,

$$q^{-1} = q^{*}/||q||^{2} = (a - b\mathbf{i} - c\mathbf{j} - d\mathbf{k})/(a^{2} + b^{2} + c^{2} + d^{2})$$

Powers

Complex numbers can be represented in polar form:

$$p = (a + b\mathbf{i}) = ||p||e^{\mathbf{i}\theta} = ||p||(\cos\theta + \mathbf{i}\sin\theta)$$

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$$q = ||q||e^{\mathbf{n}\theta} = ||q||(\cos\theta + \mathbf{n}\sin\theta)$$

where **n** is a unit, pure quaternion vector. From DeMoivre's formula it follows that

$$q^t = ||q||^t e^{t\mathbf{n}\theta} = ||q||^t(\cos t\theta + \mathbf{n}\sin t\theta)$$

Rotation

- Euler angles - combination of 3 independent rotations
- Rotation matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

  - Composition through multiplication - NOT commutative - order is important
  - Rotation sequence $R_1, R_2, \ldots R_n$ represented by product $R_n \ldots R_2 R_1$
  - Transform $\mathbf{v}$ by R: $\mathbf{v}' = R\mathbf{v}$.

Rotation

- Quaternion - mapping from rotations to unit quaternions

$$q = (\cos\frac{\theta}{2} + \mathbf{n}\sin\frac{\theta}{2})$$

  where $\mathbf{n}$ is the unit vector direction of the axis of rotation. θ is the amount of rotation about the axis.
- Composition: $q_n \cdots q_2 q_1$
- Transform $\mathbf{v}$ by q: $\mathbf{v}' = q\mathbf{v}q^{-1}$

Slerp

Spherical Linear IntERPolation

- Geometric

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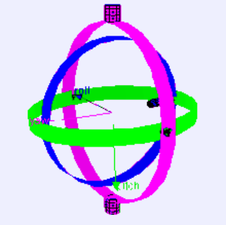
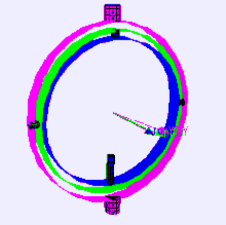
  - Ken Shoemake, *Animating Rotation with Quaternion Curves*, SIGGRAPH 1985

Gimbal Lock



Gimbals



Gimbal Lock

Image Source — Wikipedia

Normalization

- Problem with accumulating small rotations

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  - numerical error

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  - numerical error
- Matrix form - re-orthogonalize, renormalize rows or columns
- Quaternion form - simply renormalize

Rotational Hysteresis

- Rotational hysteresis - final orientation dependent on path taken
- Ken Shoemake, *ARCBALL: a user interface for specifying three-dimensional orientation using a mouse*, 1992
  - No rotational hysteresis
  - quaternion based

Conclusion

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