Cyclic CQ scheme for handling multiple dose-volume constraints in inverse planning of Intensity-Modulated Photon or Proton Therapy

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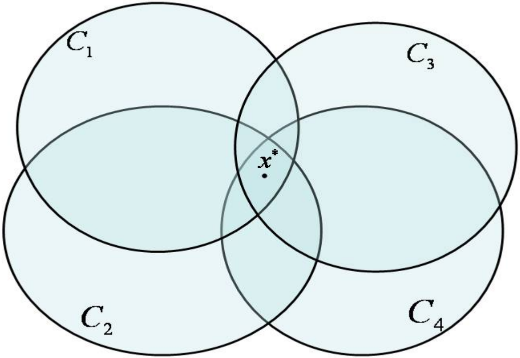
* Joint work with Mark Brooke, Yair Censor, Scott Penfold, Reinhard Schulte and Frank Van Den Heuvel

The Convex Feasibility Problem (CFP)

Given m nonempty, closed and convex sets $C_i \subset \mathbb{R}^n$, the CFP is formulated as follows:

$$\text{find a point } x^* \in \cap_{i=1}^m C_i \neq \emptyset. \tag{1}$$



Linear Convex Feasibility Problem

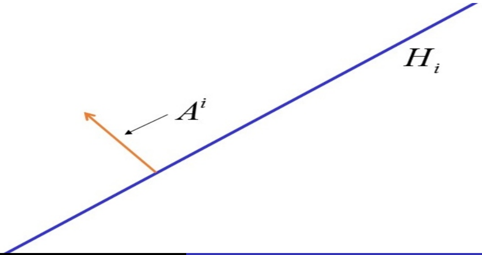
Consider the system of linear equations represented by

$$Ax = b \tag{2}$$

where $A \in \mathcal{M}at_{m\times n}(\mathbb{R})$, $x \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$.

Denote by A^i the i-th row of A, the corresponding entries b_i of b, and

$$H_i := \{x \in \mathbb{R}^n \mid \langle A^i, x\rangle = b_i\}. \tag{3}$$

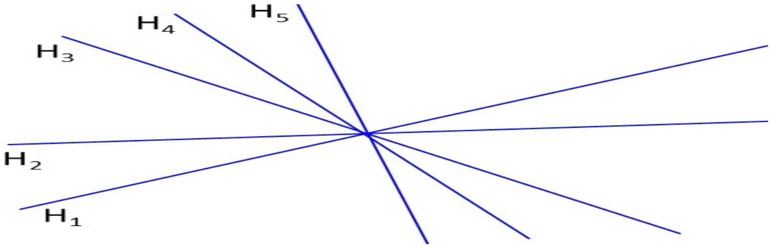


Linear Convex Feasibility Problem

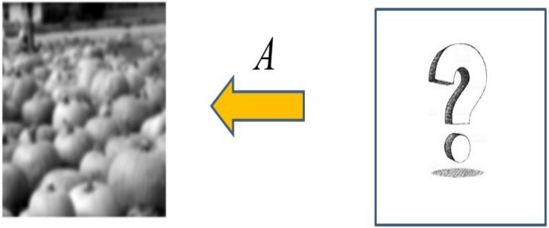
$$\boxed{Ax^* = b}$$

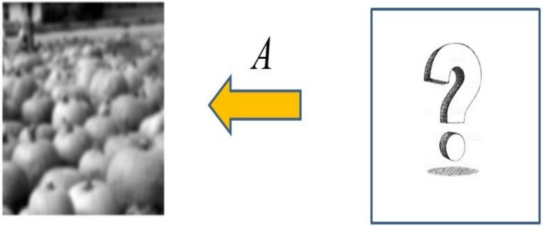
$$\Updownarrow$$

$$\boxed{x^* \in \bigcap_{i=1}^{m} H_i}$$



Applications - Image Recovery


A



$$b = \begin{bmatrix} 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&8&8&0&0&0&0&4&4&0&0&0&0&0&0&0 \\ 0&8&8&0&0&0&0&4&4&0&3&3&3&3&3&0 \\ 0&8&8&0&0&0&0&4&4&0&3&3&3&3&3&0 \\ 0&8&8&0&0&0&0&4&4&0&3&3&3&3&3&0 \\ 0&8&8&0&0&0&0&4&4&0&3&3&3&3&3&0 \\ 0&8&8&8&8&8&0&4&4&0&3&3&3&3&3&0 \\ 0&8&8&8&8&8&0&4&4&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \end{bmatrix}$$

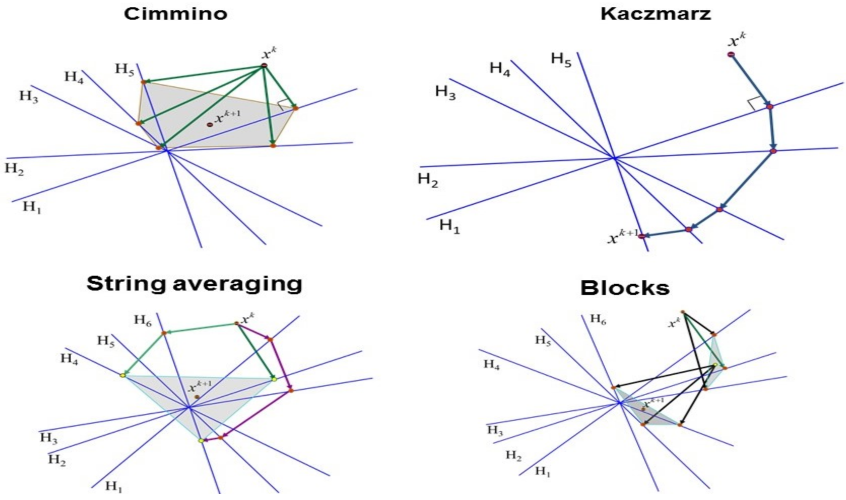
$Ax = b$

Or

$\min_{x \in \mathbb{R}^n} \|Ax - b\|^2$

$= x$

Projection methods for solving $Ax = b$

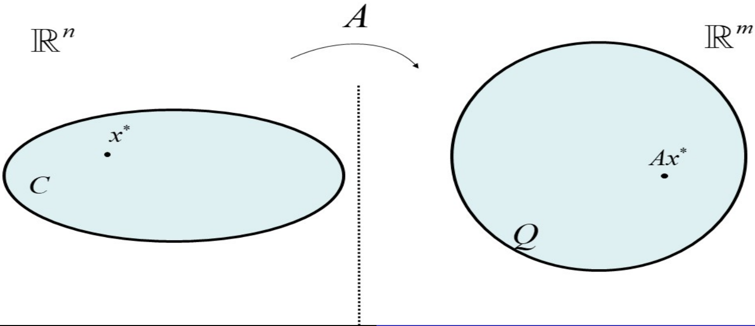


The figure is reproduced from [3].

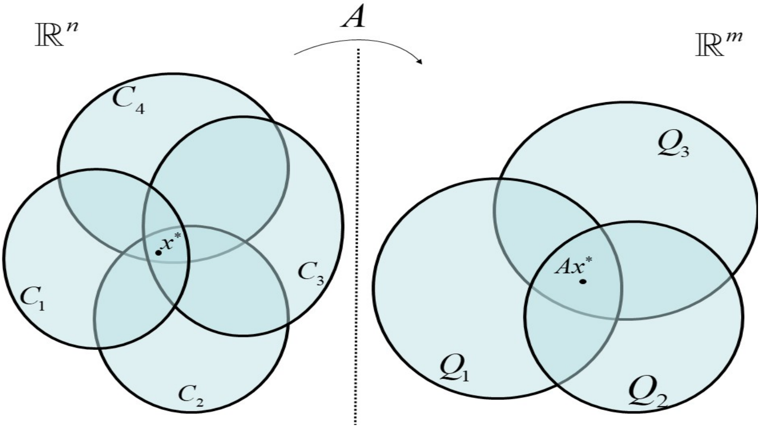
The Split Convex Feasibility Problem (SCFP)

Given two nonempty, closed and convex sets $C \subset \mathbb{R}^n$ and $Q \subset \mathbb{R}^m$ and a matrix $A \in \mathcal{M}at_{m\times n}(\mathbb{R})$, the SFP is formulated as follows:

$$\text{find a point } x^* \in C \text{ and } Ax^* \in Q. \tag{4}$$

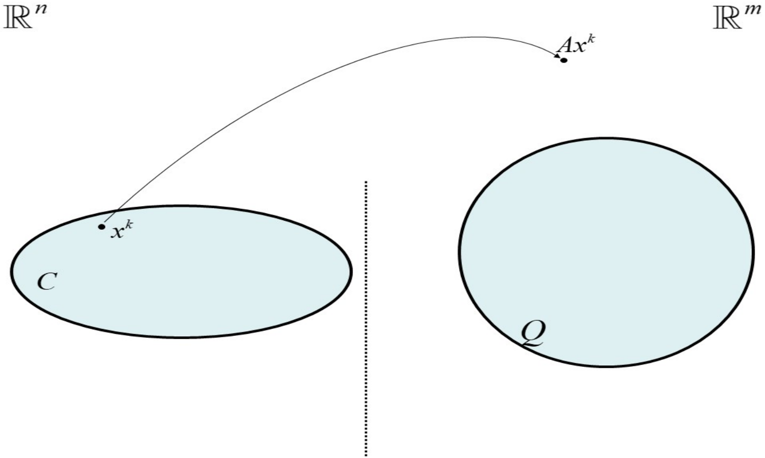


The Multiple-Sets Split Feasibility Problem



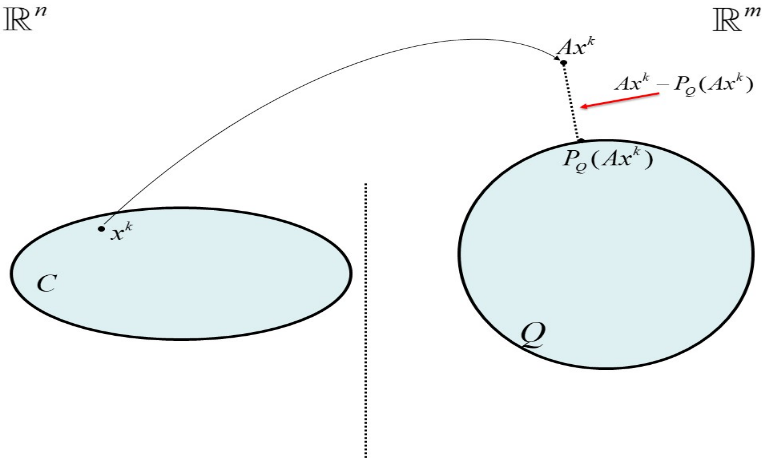
Byrne CQ algorithm for solving SCFPs

$$x^{k+1} = P_C\left(x^k - \gamma A^T\left(Ax^k - P_Q\left(Ax^k\right)\right)\right) \tag{5}$$



Byrne CQ algorithm for solving SCFPs

$$x^{k+1} = P_C\left(x^k - \gamma A^T\left(Ax^k - P_Q\left(Ax^k\right)\right)\right) \quad (6)$$



Byrne CQ algorithm for solving SCFPs

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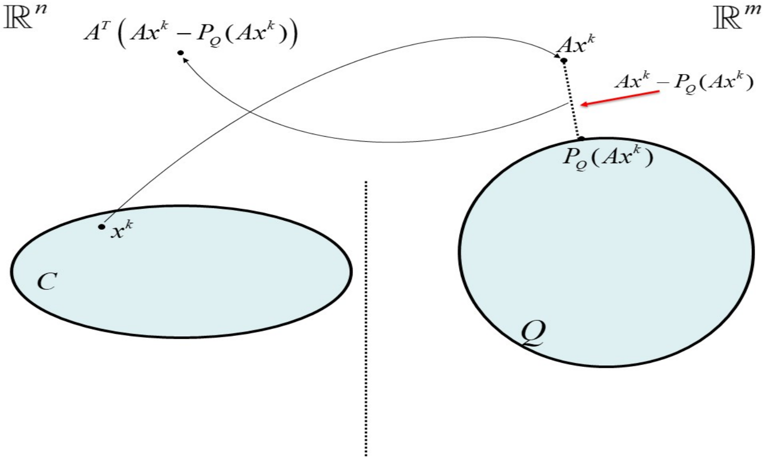
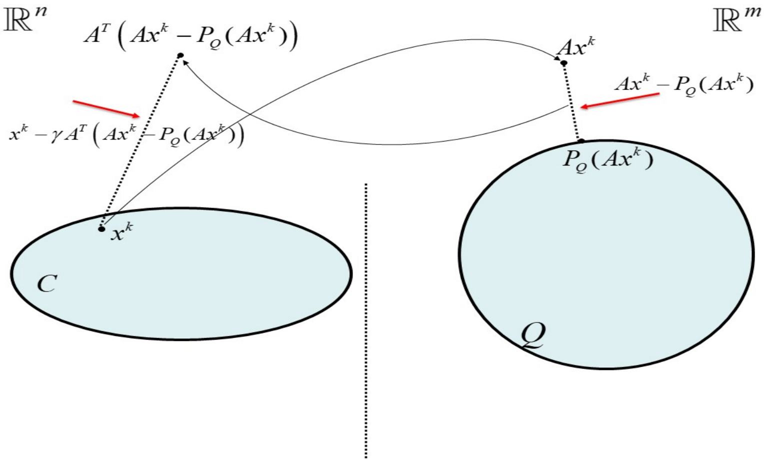


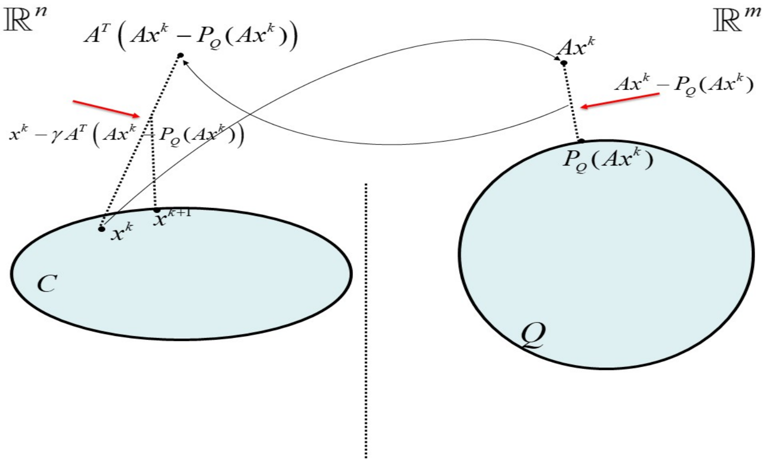
Byrne CQ algorithm for solving SCFPs

$$x^{k+1} = P_C\left(x^k - \gamma A^T\left(Ax^k - P_Q\left(Ax^k\right)\right)\right) \tag{8}$$



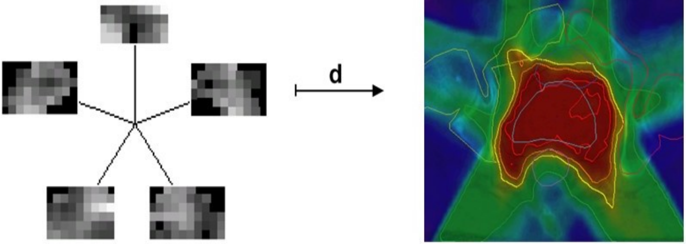
Byrne CQ algorithm for solving SCFPs

$$x^{k+1} = P_C\left(x^k - \gamma A^T\left(Ax^k - P_Q\left(Ax^k\right)\right)\right) \qquad (9)$$



Intensity-Modulated Radiation Therapy (IMRT)

Treatment plan | **Dose distribution**



intensity vector $\mathbf{x} \geq 0$ over $\sim 10^3$ beamlets

dose vector $\mathbf{d}(\mathbf{x})$ over $\sim 10^6$ voxels V_j

Equivalent Uniform Dose (EUD)-based IMRT

1. $\mathbb{R}^J-$ radiation intensity space.
2. $\mathbb{R}^I-$ the dose space.
3. $x = (x_j)_{j=1}^J \in \mathbb{R}^J$ the intensities vector.
4. $a_{ij} \geq 0$ denotes the dose absorbed in voxel i due to radiation of unit intensity from the j-th beamlet.
5. $d = (d_i)_{i=1}^I \in \mathbb{R}^I$ denote the dose vector, where its entries, d_i represent the total dose absorbed in voxel i, therefore

   $d_i = \Sigma_{j=1}^J a_{ij} x_j$.

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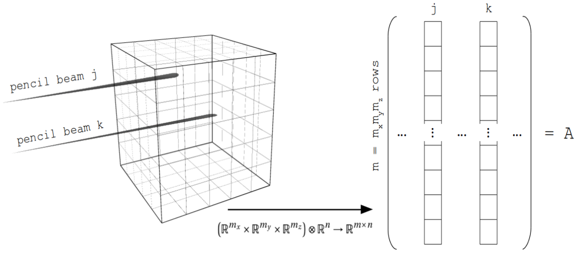


Intensity-Modulated Proton Therapy (IMPT)

A fully-discretized modeling approach for inverse radiation therapy treatment planning leads to the following linear feasibility problem.

$$c \leq Ax \leq b, \tag{10}$$

where A is the dose matrix, x is the unknown vector of intensities and the vectors b and c contain upper and lower bounds on the total dose Ax permitted and required in volume elements (voxels) of sensitive organs/tissues and target areas, respectively, inside the irradiated body.

Dose Volume Constraints (DVCs)

Consider a subset of size m_1 of the linear inequalities described by $c \leq Ax \leq b$ before.

$$A_1 x \leq b^1 \tag{11}$$

Now, up to a $\alpha\%$ of m_1, the right (left) hand side bounds may be potentially violated by up to a fraction $\beta\%$ of their values. This results in.

$$A_1 x \leq (1+\beta) b^1 \tag{12}$$

and

$$\|(Ax - b^1)_+\|_0 \leq \alpha m_1 . \tag{13}$$

where for $z \in \mathbb{R}^d$, $\|(z)_+\|_0 := |\{z_i \mid z_i > 0\}|$.

Split **NON-CONVEX** feasibility problem

$$\text{find } x \in C \text{ such that } A_1 x \in Q. \tag{14}$$

where

$$C := \{x \in \mathbb{R}^n \mid c \leq Ax \leq b\} \cap \mathbb{R}^n_+ \tag{15}$$

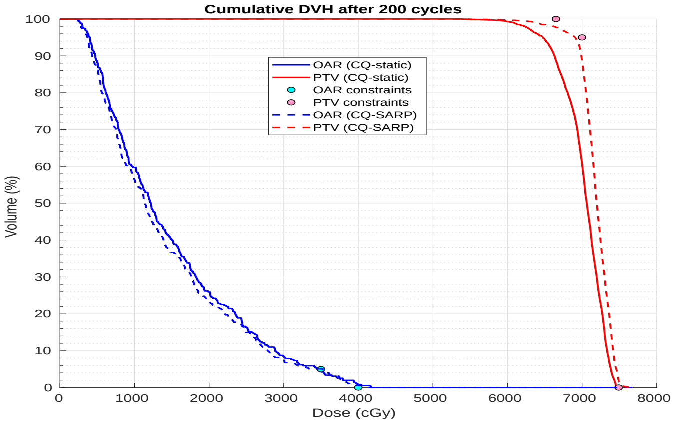
and

$$Q := \left\{ y \in \mathbb{R}^{m_1} \mid \left\| \left(y - b^1 \right)_+ \right\|_0 \leq \alpha m_1 \right\}. \tag{16}$$

Apply CQ algorithm and the *Automatic Relaxation Method* (ARM) is used to approximate P_C.

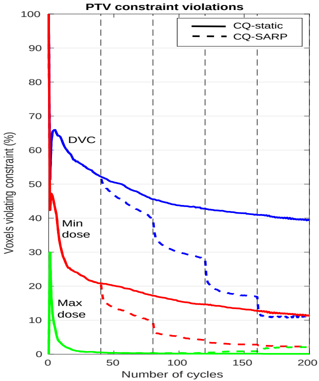
Results

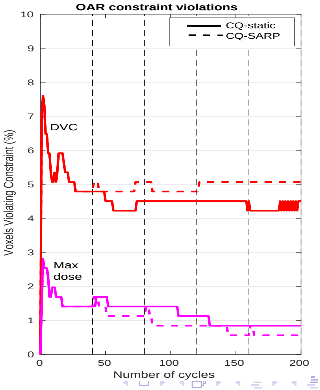
Dose-Volume Split Feasibility algorithm with two DVCs, one on the OAR (brainstem) and one on the PTV. **CQ-SARP** refers to the use of the Self-Adaptive Relaxation Parameter method. The ARM is used for the projections onto C.



Results

The percentage of voxels violating their constraints are shown here as a function of the number of algorithm cycles. The dotted lines show that the convergence rate generally increases using the CQ-SARP method. The vertical dotted reference lines indicate the cycles at which the algorithm's performance is evaluated and the parameters are updated accordingly.





- Given $C_i \subseteq \mathbb{R}^n$, $1 \leq i \leq p$ and $Q_j \subseteq \mathbb{R}^m$, $1 \leq j \leq r$, are closed and convex sets.
- $A_j \in \mathbb{R}^{m \times n}$, for $1 \leq j \leq r$.
- Let $\Omega \subseteq \mathbb{R}^n$ be another convex set.

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Masad and Reich *Constrained Multiple Set Split Convex Feasibility Problem* (CMSSCFP) consists of finding a point $x^* \in \Omega$ such that

$$\text{find } x^* \in \cap_{i=1}^{p} C_i \text{ such that } A_j x^* \in Q_j. \tag{17}$$

We propose a self-adaptive string-averaging CQ-algorithm, in particular the following cyclic CQ variant for **CONVEX** sets:

$$x^{k+1} = P_{C[k]}(x^k - \gamma_{[k]} A_{[k]}^T (A_{[k]} x^k - P_{Q[k]}(A_{[k]} x^k))) \tag{18}$$

where $\{\gamma_{[k]}^k\}_{k \geq 0} \subseteq \left(0, \frac{2}{\|A_{[k]}\|}\right)$ and $[k] := k \bmod p$.

*Enable to replace $\|(\cdot)_+\|_0$ with (convex) l1 approximation $\|z\|_1 := \sum_{i=1}^{d} |z_i|$.

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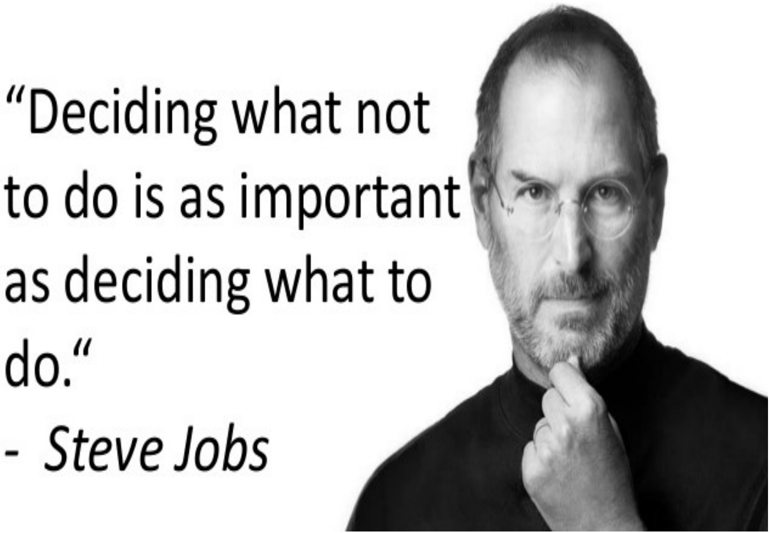
Mathematical theory for the CQ algorithm which is not restricted to convex setting. In case of multiple sets in each space the method has a simultaneous nature, for example:

$$x^{k+1} = P_C\left(x^k - \gamma \sum_{j=1}^{r} A_j^T(A_j x^k - P_{Q_j}(A_j x^k))\right) \tag{19}$$


"Deciding what not to do is as important as deciding what to do."
- Steve Jobs


THANK
YOU
GRACIAS
ARIGATO
SHUKURIA
DANKSCHEEN
BIYAN
SHUKRIA
TINGKI
YAQHANYELAY
TASHAKKUR ATU
SUKSAMA
EKHMET
GRAZIE
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KOMAPSUMNIDA
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